

**An Initial and boundary value problem on a strip
for a large class of quasilinear hyperbolic systems arising
from an atmospheric model.**

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Abstract. In this paper well-posedness is proved for an initial and boundary value problem (IBVP) relative to a large class of quasilinear hyperbolic systems, in $p + q$ equations, on a strip, arising from a model of H_2O -phase transitions in the atmosphere. To obtain this result, first, we extensively study an IBVP for the generic linear transport equation on $S_{t_1} = (0, t_1) \times G$ with uniformly locally Lipschitz data and associated vector field in $L_{x_d}^-(S_{t_1} \times \mathbb{R}^p)$ (this cone of $L_t^\infty(0, t_1; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p))^d$ is not contained in $W_{(t, x, y_{loc})}^{1, \infty}(S_{t_1} \times \mathbb{R}^p)^d$), that involves parametric vector functions in $L^\infty(0, t_1; W^{1, \infty}(G))^p$, by the method of characteristics. We obtain that the solution belongs to $W_{(t, x, y_{loc})}^{1, \infty}(S_{t_1} \times \mathbb{R}^p)$ and interesting estimates about it.

Afterwards, using fixed point arguments, we establish the local existence, in time, and uniqueness of the solution in $W^{1, \infty}(S_{t_*})^{p+q}$ for our class of quasilinear hyperbolic systems. Finally, we apply this result to study an IBVP for the hyperbolic part of an atmospheric model on the transition of water in the three states, introduced in [28], such that rain and ice fall from it.

Key words : initial and boundary value problem, quasilinear hyperbolic system, method of characteristics, phase transitions.

MSC : 35Q35, 35L60, 76T30, 76N10 , 35L45, 76T10.

1 - Introduction.

In [28] we introduced and studied a model of motion of the air and the phase transitions of water in the atmosphere. The purpose of this research was to provide a detailed mathematical description about the phenomena which occur in the atmosphere such as wind or cloud formation and to show its consistency. In order to obtain some mathematical results, we introduced several simplifications in our model. One of these assumed that the velocities of gas, water droplets and ice crystals are tangent to the boundary of a fixed spatial domain; therefore, we excluded, from our study, rain and snowfall. Hence,

in the paper that follows, we include these phenomena limiting the study of the model to its hyperbolic part.

More precisely, the aim of the present paper is to establish a theorem of well-posedness, in $W^{1,\infty}$, of an initial and boundary value problem (IBVP) for a large class of quasilinear hyperbolic systems defined on a strip. There is an important application of this theorem, as we will see in the last section, to an IBVP arising from a model of water phase transitions in the atmosphere given in [28]. Indeed, in the last section, we will obtain a result of well-posedness for the hyperbolic part of the atmospheric model defined in a strip for which rain and ice fall from the strip

Of course, to study this class of quasilinear hyperbolic systems, we use a linearization procedure. Therefore we need to have a result on the IBVP for the linear transport equation defined on a strip that can be applied to our quasilinear hyperbolic system. The IBVP for the linear transport equation was studied in [3] by C. Bardos assuming time independent and Lipschitz vector field and using the method of characteristics and the semi-group theory. Hence, we can not use this result because the vector field of our system is depending on time.

Instead, in the paper [9], F. Boyer gives the trace theorems for the weak solutions of the linear transport equation on a regular bounded domain Ω and solves the corresponding IBVP, assuming in particular the vector field in $L^1(0, T; W^{1,p}(\Omega))$ with null divergence and its trace in $L^p((0, T) \times \partial\Omega)$ ($p > 1$). Furthermore, O. Besson and J. Pousin study in [8] an IBVP for the linear transport equation on a bounded set assuming a L^∞ -vector field with L^∞ -divergence. They use a particular functional setting of an anisotropic Sobolev space and moreover the IBVP is reformulated by time-space least squares. Afterwards, G. Crippa, C. Donadello and L. V. Spinolo, establish in [18] well-posedness for continuity equations with bounded total variation coefficients; in particular, the vector field is bounded with bounded divergence and it is in $L^1_{loc}(0, T; BV(\Omega))$. Unfortunately, in [9], [8] and [18], the domain is bounded and the vector field is also less regular to obtain estimates about the gradient of the solution for the linear transport equation, therefore we can not use these results to study our IBVP for the quasilinear hyperbolic systems. Hence, using a more regular vector field depending also on time, the definition of generalized solution given in [27] and the method of characteristics, we obtain a result on well-posedness in $W^{1,\infty}$ of the IBVP for the linear transport equation with parametric vector and source functions on a strip. Moreover, from this result we deduce some useful estimates that play a vital role to study the analogous IBVP for our class of quasilinear hyperbolic systems.

Let us say something about the sections of this paper. In section 2, we define some notations and functional spaces that will be used in that follows. In section 3, we introduce an IBVP for a large class of quasilinear hyperbolic systems and we transform this problem into system of integral equations of Volterra type using an extension of Cinquini-Cibrario method of characteristics. Therefore a generalized solution of our IBVP will be a solution of this system of integral equations. Hence, we give the statement of the main theorem regarding the well-posedness for the given IBVP about generalized solutions. In section 4, we introduce an analogous IBVP for a linear transport equation with vector parametric and source functions. In section 5, we study in detail the flow associated to the vector field relative to the linear transport equation, obtaining results about the regularity of the flow and the initial time of existence of the flow. From these results, we are able, in section 6, to prove the well-posedness of the IBVP for the linear transport equation that includes a parametric vector function.

In section 7, after having obtained a preliminar result of existence and uniqueness about the semilinear part of our quasilinear hyperbolic system, finally, we prove the main theorem of this paper.

The paper ends (section 8) with the application of the main theorem to an IBVP for the hyperbolic part of an atmospheric model.

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2 - Some useful notations and functional spaces.

In this section we define some notations and functional spaces that will be used later. First of all, let G be an open subset of \mathbb{R}^d defined as follows

$$(2.1) \quad G = \mathbb{R}^{d-1} \times (0, 1).$$

The canonical frame, in the linear space \mathbb{R}^d , is denoted by $\{e_j | j = 1, \dots, d\}$. Moreover, we introduce

$$(2.2) \quad S_{t_1} = (0, t_1) \times G, \quad S'_{t_1} = (0, t_1) \times \mathbb{R}^{d-1},$$

where $t_1 > 0$. The generic point of S_{t_1} is denoted by $(t, x) = (t, x', x_d)$.

The maximum and minimum value of the real numbers α and β is indicated by $\alpha \vee \beta$ and $\alpha \wedge \beta$ respectively; furthermore we define

$$(2.3) \quad I(\alpha, \beta) = (\alpha \wedge \beta, \alpha \vee \beta) \quad \forall \alpha, \beta \in \mathbb{R}.$$

Now, we introduce the following functional spaces

$$(2.4) \quad L_t^r(t_2, t_3; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p)) = \bigcap_{A>0} L^r(t_2, t_3; L^\infty(G \times (-A, A)^p)),$$

$$(2.5) \quad L_t^r(t_2, t_3; W_{(x, y_{loc})}^{1,\infty}(G \times \mathbb{R}^p)) = \bigcap_{A>0} L^r(t_2, t_3; W^{1,\infty}(G \times (-A, A)^p)),$$

where $0 \leq t_2 \leq t_3 \leq t_1$, $1 \leq r \leq \infty$ and $p > 0$ is an integer. Moreover, assuming $A > 0$ and $0 \leq a < b \leq 1$, we say that a measurable function $g : S_{t_1} \times \mathbb{R}^p \rightarrow \mathbb{R}$ belongs to $L_{x_d}^r(a, b; W_{(t, x', y)}^{1,\infty}(S'_{t_1} \times (-A, A)^p))$ if and only if

$$(2.6) \quad \|g\|_{L_{x_d}^r(a, b; W_{(t, x', y)}^{1,\infty}(S'_{t_1} \times (-A, A)^p))} = \int_a^b \|g(\cdot, z, \cdot)\|_{W^{1,\infty}(S'_{t_1} \times (-A, A)^p)}^r dz < \infty;$$

therefore we can define

$$(2.7) \quad L_{x_d}^r(a, b; W_{(t, x', y_{loc})}^{1,\infty}(S'_{t_1} \times \mathbb{R}^p)) = \bigcap_{A>0} L_{x_d}^r(a, b; W_{(t, x', y)}^{1,\infty}(S'_{t_1} \times (-A, A)^p)).$$

Furthermore the product of k copies, for example, of $L_t^r(t_2, t_3; W_{(x, y_{loc})}^{1,\infty}(G \times \mathbb{R}^p))$ will be denoted as follows

$$(2.8) \quad L_t^r(t_2, t_3; W_{(x, y_{loc})}^{1,\infty}(G \times \mathbb{R}^p))^k,$$

where $k > 0$ is an integer.

To avoid long estimates, we must introduce shorter symbols to indicate some norms. For example, we often consider a (vector) function $h : S_{t_1} \rightarrow \mathbb{R}^p$ (or \bar{h} , or $h^{(l)}$, etc) with a given regularity and, in short, we put

(2.9)

$$\|h\|_\infty = \|h\|_{L^\infty(S_{t_1})}, \quad \|\partial_{x_j} h\|_\infty = \left\| \frac{\partial h}{\partial x_j} \right\|_{L^\infty(S_{t_1})}, \quad \|D_{(t,x)} h\|_\infty = \|D_{(t,x)} h\|_{L^\infty(S_{t_1})},$$

where $j = 1, \dots, d$ and $D_{(t,x)} h$ is the jacobian matrix for h . Furthermore, if α and $\bar{\alpha}$ (vector or matrix functions on S_{t_1}) have bounded components, then we put

$$(2.10) \quad \Lambda(\alpha, \bar{\alpha}) = 1 + \|\alpha\|_\infty \vee \|\bar{\alpha}\|_\infty.$$

In many situations, we consider a function $g \in L_t^r(t_2, t_3; W_{(x, y_{loc})}^{1, \infty}(G \times \mathbb{R}^p))$, two bounded functions $h, \bar{h} : S_{t_1} \rightarrow \mathbb{R}^p$ and we must study the following term

$$(2.11) \quad I = \left\| \operatorname{ess\,sup}_{x \in G} g(\cdot, x, h(\cdot, x)) \vee \operatorname{ess\,sup}_{x \in G} g(\cdot, x, \bar{h}(\cdot, x)) \right\|_{L^r(t_2, t_3)},$$

where $\operatorname{ess\,sup}$ is the essential supremum. It is obvious that we have

$$(2.12) \quad I \leq \|g\|_{L^r(t_2, t_3; L^\infty(G \times (-\|h\|_\infty \vee \|\bar{h}\|_\infty, \|h\|_\infty \vee \|\bar{h}\|_\infty)^p))};$$

as well, for convenience, we introduce the following abbreviated notation

$$(2.13) \quad \|g\|_{L^r(t_2, t_3; h, \bar{h})} = \|g\|_{L^r(t_2, t_3; L^\infty(G \times (-\|h\|_\infty \vee \|\bar{h}\|_\infty, \|h\|_\infty \vee \|\bar{h}\|_\infty)^p))}, \quad 1 \leq r \leq \infty.$$

In a similar way, we define

$$(2.14) \quad \|\cdot\|_{L^\infty(G; h, \bar{h})} = \|\cdot\|_{L^\infty(G \times (-\|h\|_\infty \vee \|\bar{h}\|_\infty, \|h\|_\infty \vee \|\bar{h}\|_\infty)^p)},$$

$$(2.15) \quad \|\cdot\|_{L^\infty(S'_{t_1}; h, \bar{h})} = \|\cdot\|_{L^\infty(S'_{t_1} \times (-\|h\|_\infty \vee \|\bar{h}\|_\infty, \|h\|_\infty \vee \|\bar{h}\|_\infty)^p)},$$

$$(2.16) \quad \|\cdot\|_{L_{x_d}^r(a, b; h, \bar{h})} = \|\cdot\|_{L_{x_d}^r(a, b; L^\infty(S'_{t_1} \times (-\|h\|_\infty \vee \|\bar{h}\|_\infty, \|h\|_\infty \vee \|\bar{h}\|_\infty)^p))}.$$

Of course, if $h = \bar{h}$, we can simplify the notations assuming

$$(2.17) \quad \|\cdot\|_{L^r(t_2, t_3; h)} = \|\cdot\|_{L^r(t_2, t_3; h, h)}, \quad \text{etc.}$$

Moreover, assuming K a subset of \bar{S}_{t_1} , we denote the family of uniformly locally lipschitz functions on K by $Lip_{loc}^{unif}(K)$, i.e.

$$(2.18) \quad Lip_{loc}^{unif}(K) = \{\phi : K \rightarrow \mathbb{R} \mid \exists M > 0 : \forall (t, x) \in K \quad \exists \sigma > 0 :$$

$$\forall (t', x') \in K, \quad |t - t'| + |x - x'| \leq \sigma \Rightarrow |\phi(t, x) - \phi(t', x')| \leq M [|t - t'| + |x - x'|]\}.$$

Finally, it is important in what follows to consider, in $L_t^\infty(0, t_1; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p))^d$, the following cone

(2.19)

$$L_{x_d}^-(S_{t_1} \times \mathbb{R}^p) = \left\{ b : S_{t_1} \times \mathbb{R}^p \rightarrow \mathbb{R}^d \mid b(t, x, y) = (b'(t, x, y), b_d(t, x)) \text{ a.e. } (t, x, y) \in S_{t_1} \times \mathbb{R}^p, \right.$$

$$b \in L_t^\infty(0, t_1; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p))^d \cap L_t^1(0, t_1; W_{(x, y_{loc})}^{1, \infty}(G \times \mathbb{R}^p))^d, \quad b_d \in L_{x_d}^1(0, 1; W_{(t, x')}^{1, \infty}(S'_{t_1})),$$

$$\exists B_d > 0 : b_d(t, x) \leq -B_d \quad \text{a.e. } t \in (0, t_1), \quad \forall x \in G \Big\}.$$

Remark 2.0.0. We observe that if $b \in L_{x_d}^-(S_{t_1} \times \mathbb{R}^p)$ then b_d can not belong to $W^{1, \infty}(S_{t_1})$. Indeed, assuming, for example $d = 1$, we have

$$(x_1 - t - 1)^{1/3} - 1 \in L_{x_1}^-(S_{t_1} \times \mathbb{R}^p) \setminus W^{1, \infty}(S_{t_1}).$$

3 - Position of the problem and main theorem.

In this paper we study the IBVP for a large class of quasilinear hyperbolic systems in a strip, that generalizes the hyperbolic part, without integral terms, of a model of water phase transitions in the atmosphere given in [28] (see also Section 8). More precisely, we study, in the unknown functions $y = (y_1, \dots, y_p)$, $w = (w_1, \dots, w_q)$, the following IBVP

$$(3.1) \quad \partial_t y_i(t, x) + v_i(t, x) \cdot \nabla_x y_i(t, x) = f_i(t, x, y(t, x), w(t, x)), \quad (t, x) \in S_{t_1},$$

$$(3.2) \quad \begin{aligned} \partial_t w_j(t, x) + u'_j(t, x, y) \cdot \nabla_{x'} w_j(t, x) + u_{jd}(t, x) \partial_{x_d} w_j(t, x) = \\ = g_j(t, x, y(t, x), w(t, x)), \quad (t, x) \in S_{t_1}, \end{aligned}$$

$$(3.3) \quad y_i(0, x) = y_{0i}(x), \quad x \in G,$$

$$(3.4) \quad w_j(t, x) = w_j^*(t, x), \quad (t, x) \in \Gamma_-,$$

where $i = 1, \dots, p$, $j = 1, \dots, q$,

$$(3.5) \quad \Gamma_- = (\{0\} \times G) \cup ([0, t_1] \times \mathbb{R}^{d-1} \times \{1\}),$$

$$(3.6) \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_{x_k} = \frac{\partial}{\partial x_k}, \quad \nabla_x = (\partial_{x_1}, \dots, \partial_{x_d}), \quad \nabla_{x'} = (\partial_{x_1}, \dots, \partial_{x_{d-1}}), \quad k = 1, \dots, d.$$

Moreover $f_i, g_j : S_{t_1} \times \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$ are given sources for the corresponding transport equations, $v_i : S_{t_1} \rightarrow \mathbb{R}^d$ and $u_j = (u_{j1}, u_{j2}, \dots, u_{jd-1}, u_{jd}) = (u'_j, u_{jd}) : S_{t_1} \times \mathbb{R}^p \rightarrow \mathbb{R}^d$ are vector fields, $y_{0i} : G \rightarrow \mathbb{R}$, $w_j^* : \Gamma_- \rightarrow \mathbb{R}$ are given functions and $i = 1, \dots, p$, $j = 1, \dots, q$.

Afterwards, we can associate to the IBVP (3.1)-(3.4), by the method of characteristics, the following system of integral equations

$$(3.7) \quad y_i(t, x) = y_{0i}(X_{iY}(0; t, x)) + \int_0^t f_i(s, X_{iY}(s; t, x), y(s, X_{iY}(s; t, x), w(s, X_{iY}(s; t, x))) ds,$$

$$(3.8) \quad w_j(t, x) = w_j^*(\tau_{j-}(t, x, y), X_{jW}(\tau_{j-}(t, x, y); t, x, y)) +$$

$$+ \int_{\tau_{j-}(t,x,y)}^t g_j(s, X_{jW}(s; t, x, y), y(s, X_{jW}(s; t, x, y)), w(s, X_{jW}(s; t, x, y))) ds,$$

where $i = 1, \dots, p$, $j = 1, \dots, q$, X_{iY} and X_{jW} are the fluxes associated to the vector fields v_i and w_j respectively and, therefore, are defined as follows

$$(3.9) \quad X_{iY}(s; t, x) = x - \int_s^t v_i(r, X_{iY}(r; t, x)) dr, \quad s \in [0, t],$$

$$(3.10) \quad X_{jW}(s; t, x, y) = x - \int_s^t u_j(r, X_{jW}(r; t, x, y), y(r, X_{jW}(r; t, x, y))) dr, \quad s \in [\tau_{j-}(t, x, y), t],$$

whereas $\tau_{j-}(t, x, y)$ is the minimal time of existence for the solution of (3.10).

Therefore, we assume that a solution for the system (3.7) and (3.8) (see Theorem 3.1) is a generalized solution for the IBVP (3.1)-(3.4). This definition is consistent with the one given by A. Myshkis in [27].

Hence, we assume the following conditions on the functions that appear in the IBVP (3.1)-(3.4) :

$$(3.11) \quad f_i, g_j \in L_t^\infty(0, t_1; L_{(x, y_{loc}, w_{loc})}^\infty(G \times \mathbb{R}^p \times \mathbb{R}^q)) \cap L_t^1(0, t_1; W_{(x, y_{loc}, w_{loc})}^{1, \infty}(G \times \mathbb{R}^p \times \mathbb{R}^q)),$$

$$(3.12) \quad v_i \in L^\infty(0, t_1; L^\infty(G))^d \cap L^1(0, t_1; W^{1, \infty}(G))^d,$$

$$v_i(t, x', 0) \cdot e_d = v_i(t, x', 1) \cdot e_d = 0 \quad \forall x' \in \mathbb{R}^{d-1}, \text{ a.e. } t \in (0, t_1),$$

$$(3.13) \quad u_j \in L_{x_d}^-(S_{t_1} \times \mathbb{R}^p),$$

$$(3.14) \quad y_{0i} \in W^{1, \infty}(G), \quad w_j^* \in Lip_{loc}^{unif}(\Gamma_-), \quad i = 1, \dots, p, \quad j = 1, \dots, q.$$

Now, we are ready to give the main theorem of this paper

Theorem 3.1. *Assume that the hypotheses (3.11)-(3.14) are verified. Then there exists $0 < t^* \leq t_1$ such that the system of integral equations (3.7) and (3.8), where X_{iY} and X_{jW} are given by (3.9) and (3.10), admits one and only one solution $(y, w) \in W^{1, \infty}(S_{t^*})^{p+q}$. The vector function (y, w) is also said to be the generalized solution for IBVP (3.1)-(3.4).*

Moreover, for any sufficiently small t , the mapping $(y_0, w^*, v_1, \dots, v_p, u_1, \dots, u_q, f, g) \in W^{1, \infty}(G)^p \times Lip_{loc}^{unif}(\Gamma_-)^q \times (L^\infty(0, t; L^\infty(G)) \cap L^1(0, t; W^{1, \infty}(G)))^{dp} \times L_{x_d}^-(S_t \times \mathbb{R}^p)^q \times [L_t^\infty(0, t; L_{(x, y_{loc}, w_{loc})}^\infty(G \times \mathbb{R}^p \times \mathbb{R}^q)) \cap L_t^1(0, t; W_{(x, y_{loc}, w_{loc})}^{1, \infty}(G \times \mathbb{R}^p \times \mathbb{R}^q))]^{p+q} \rightarrow (y, w) \in L^\infty(S_t)^{p+q}$ is locally Lipschitz continuous.

To prove this theorem we must first deduce some results about the linear transport equations.

4 - An initial and boundary value problem for the linear transport equation with parametric vector function and source term.

In this section we study the following linear transport equation

$$(4.1) \quad \partial_t z(t, x) + b(t, x, h(t, x)) \cdot \nabla_x z(t, x) + c(t, x, h(t, x)) z(t, x) = a(t, x, h(t, x)), \quad (t, x) \in S_{t_1},$$

under the following condition

$$(4.2) \quad z(t, x) = z^*(t, x) \quad (t, x) \in \Gamma_-,$$

where $b = (b', b_d) : S_{t_1} \times \mathbb{R}^p \rightarrow \mathbb{R}^d$ is a vector field with $b_d : S_{t_1} \rightarrow \mathbb{R}$ (d is an integer greater than zero), $c : S_{t_1} \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $a : S_{t_1} \times \mathbb{R}^p \rightarrow \mathbb{R}$ (source) are given functions, $h : S_{t_1} \rightarrow \mathbb{R}^p$ is a parametric vector function, Γ_- is the surface carrying data defined in (3.5) and $z^* : \Gamma_- \rightarrow \mathbb{R}$ is a given function. The problem (4.1)-(4.2) is called an initial and boundary value problem (IBVP) for the linear transport equation (4.1).

This IBVP will be studied through the analysis of the flow X of the vector field b which satisfies

$$(4.3) \quad \begin{cases} \partial_t X(t; t_0, x_0, h) = b(t, X(t; t_0, x_0, h), h(t, X(t; t_0, x_0, h))) \\ X(t_0; t_0, x_0, h) = x_0 \quad (t_0, x_0) \in S_{t_1}. \end{cases}$$

In fact, we will see, under some hypotheses, that the solution z at point $(t_0, x_0) \in S_{t_1}$ is determined by the knowledge of the flow $X(t; t_0, x_0, h)$ which starts from $\tau_-(t_0, x_0, h)$ (initial time) and the value of z^* in $(\tau_-(t_0, x_0, h), X(\tau_-(t_0, x_0, h); t_0, x_0, h)) \in \Gamma_-$ (see Section 6). The Cauchy problem (4.3) is called the characteristic problem related to the transport equation (4.1). Moreover, it is equivalent, in Caratheodory theory, to the following integral equation

$$(4.4) \quad X(t; t_0, x_0, h) = x_0 - \int_t^{t_0} b(s, X(s; t_0, x_0, h), h(s, X(s; t_0, x_0, h))) ds.$$

5 - Regularity of the flow $X(\cdot; t_0, x_0, h)$ and the initial time $\tau_-(t_0, x_0, h)$.

In this section we give some useful lemmas about the flow $X(\cdot; t_0, x_0, h)$ and the initial time $\tau_-(t_0, x_0, h)$.

Lemma 5.1. *Assume for the vector field b the following regularity*

$$(5.1) \quad b \in L_t^1(0, t_1; W_{(x, y_{loc})}^{1, \infty}(G \times \mathbb{R}^p))^d$$

and suppose that there exists a positive constant B_d such that

$$(5.2) \quad b_d(t, x) \leq -B_d \quad \forall x \in G \quad \text{a.e. } t \in (0, t_1).$$

Then, for every $(t_0, x_0, h) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$, there exists a unique maximal Caratheodory's solution $X(\cdot; t_0, x_0, h) : [\tau_-(t_0, x_0, h), \tau_+(t_0, x_0, h)] \rightarrow \overline{G}$ of (4.4) with

$$(5.3) \quad (\tau_-(t_0, x_0, h), X(\tau_-(t_0, x_0, h); t_0, x_0, h)) \in \Gamma_-.$$

Furthermore, the initial time $\tau_-(t_0, x_0, h)$ satisfies the following estimate

$$(5.4) \quad 0 \leq \tau_-(t_0, x_0, h) \leq t_0 - \rho,$$

where ρ is a number defined as follows

$$(5.5) \quad 0 < \rho \leq t_0, \quad \|b\|_{L^1(t_0-\rho, t_0; h)} < x_{0d} \wedge (1 - x_{0d}).$$

In the end, given $(t_0, x_0, h) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$, the following statements hold
 – **i)** If $s, \overline{s} \in [\tau_-(t_0, x_0, h), t_0]$ then

$$(5.6) \quad |X(s; t_0, x_0, h) - X(\overline{s}; t_0, x_0, h)| \leq \int_{I(s, \overline{s})} \|b(r, \cdot)\|_{L^\infty(G; h)} dr.$$

If $(\overline{t}_0, \overline{x}_0, \overline{h}) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$ and $s \in [\tau_-(t_0, x_0, h) \vee \tau_-(\overline{t}_0, \overline{x}_0, \overline{h}), t_0 \wedge \overline{t}_0]$ then

$$(5.7) \quad |X(s; t_0, x_0, h) - X(s; \overline{t}_0, \overline{x}_0, \overline{h})| \leq \\ \leq C_1 \left(\|b\|_{L^1(I(t_0, \overline{t}_0); h, \overline{h})} + |x_0 - \overline{x}_0| + \|h - \overline{h}\|_\infty \right),$$

where

$$(5.8) \quad C_1 = \left(1 \vee \|D_{(x,y)}b\|_{L^1(0, t_1; h, \overline{h})} \right) \exp \left(\Lambda(D_x h, D_x \overline{h}) \|D_{(x,y)}b\|_{L^1(0, t_1; h, \overline{h})} \right).$$

(Λ is defined in (2.10)).

– **ii)** Let

$$(5.9) \quad b \in L_t^\infty(0, t_1; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p))^d,$$

then (5.6) and (5.7) can be replaced, respectively, by

$$(5.10) \quad |X(s; t_0, x_0, h) - X(\overline{s}; t_0, x_0, h)| \leq \|b\|_{L^\infty(0, t_1; h)} |s - \overline{s}|,$$

$$(5.11) \quad |X(s; t_0, x_0, h) - X(s; \overline{t}_0, \overline{x}_0, \overline{h})| \leq C_2 \left[|t_0 - \overline{t}_0| + |x_0 - \overline{x}_0| + \|h - \overline{h}\|_\infty \right],$$

where

$$(5.12) \quad C_2 = \left(1 \vee \|b\|_{L^\infty(0, t_1; h, \overline{h})} \vee \|D_{(x,y)}b\|_{L^1(0, t_1; h, \overline{h})} \right) \times \\ \times \exp \left(\Lambda(D_x h, D_x \overline{h}) \|D_{(x,y)}b\|_{L^1(0, t_1; h, \overline{h})} \right).$$

Proof.

The integral equation (4.4) admits one and only one maximal solution $X \in W_{loc}^{1,1}(\tau_-(t_0, x_0, h), \tau_+(t_0, x_0, h))$ (see [21]). Now, we show that X can be extended continuously on the closed interval $[\tau_-(t_0, x_0, h), \tau_+(t_0, x_0, h)]$ and therefore it also satisfies (4.4) at the end points of this interval. Indeed, assuming s_1, s_2 such that $\tau_-(t_0, x_0, h) < s_1 \leq s_2 < \tau_+(t_0, x_0, h)$, we have

$$(5.13) \quad |X(s_2; t_0, x_0, h) - X(s_1; t_0, x_0, h)| \leq \int_{s_1}^{s_2} \|b(r, \cdot)\|_{L^\infty(G; h)} dr,$$

then, thanks to a Cauchy criterion, we deduce that exists

$$(5.14) \quad \lim_{s \rightarrow \tau_-(t_0, x_0, h)^+} X(s; t_0, x_0, h) \in \overline{G}.$$

In a similar way we study the left limit of $X(s; t_0, x_0, h)$ as s approaches $\tau_+(t_0, x_0, h)$. Therefore $X(\cdot; t_0, x_0, h) \in W^{1,1}[\tau_-(t_0, x_0, h), \tau_+(t_0, x_0, h)]$. Taking into account (5.2), it is easy to get (5.3). To prove (5.4), see, for example, [21] (Remark at page 5). Furthermore, the estimate (5.6) immediately follows from (4.4).

Now, given $(t_0, x_0, h), (\bar{t}_0, \bar{x}_0, \bar{h}) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$, we consider the solutions $X(\cdot; t_0, x_0, h), X(\cdot; \bar{t}_0, \bar{x}_0, \bar{h})$ and assume $\tau_-(t_0, x_0, h) \vee \tau_-(\bar{t}_0, \bar{x}_0, \bar{h}) \leq t_0 \wedge \bar{t}_0$. The difference between the integral representations of the previous solutions (see (4.4)) gives, if $\tau_-(t_0, x_0, h) \vee \tau_-(\bar{t}_0, \bar{x}_0, \bar{h}) \leq s \leq t_0 \wedge \bar{t}_0$, the following estimate

$$(5.15) \quad |X_d(s; t_0, x_0, h) - X_d(s; \bar{t}_0, \bar{x}_0, \bar{h})| \leq |x_{0d} - \bar{x}_{0d}| + \int_{I(t_0, \bar{t}_0)} \|b_d(r, \cdot)\|_{L^\infty(G)} dr + \\ + \int_s^{t_0 \wedge \bar{t}_0} |b_d(r, X(r; t_0, x_0, h)) - b_d(r, X(r; \bar{t}_0, \bar{x}_0, \bar{h}))| dr,$$

from which we immediately deduce

$$(5.16) \quad |X_d(s; t_0, x_0, h) - X_d(s; \bar{t}_0, \bar{x}_0, \bar{h})| \leq |x_{0d} - \bar{x}_{0d}| + \|b_d\|_{L^1(I(t_0, \bar{t}_0), L^\infty(G))} + \\ + \int_s^{t_0 \wedge \bar{t}_0} \|\nabla_x b_d(r, \cdot)\|_{L^\infty(G)} |X(r; t_0, x_0, h) - X(r; \bar{t}_0, \bar{x}_0, \bar{h})| dr.$$

Analogously we obtain

$$(5.17) \quad |X'(s; t_0, x_0, h) - X'(s; \bar{t}_0, \bar{x}_0, \bar{h})| \leq |x'_0 - \bar{x}'_0| + \|b'\|_{L^1(I(t_0, \bar{t}_0); h, \bar{h})} + \\ + \|D_y b'\|_{L^1(0, t_1; h, \bar{h})} \|h - \bar{h}\|_\infty + \int_s^{t_0 \wedge \bar{t}_0} (\|D_x b'(r, \cdot)\|_{L^\infty(G; h, \bar{h})} + \|D_y b'(r, \cdot)\|_{L^\infty(G; h, \bar{h})} \|D_x h\|_\infty) \times \\ \times |X(r; t_0, x_0, h) - X(r; \bar{t}_0, \bar{x}_0, \bar{h})| dr.$$

Adding (5.16)-(5.17) and using Gronwall's lemma, we deduce (5.7).

Of course, at this point, using again (5.17), it is not so difficult to show **ii**).

□

Now, assuming a more regular vector field b , we can obtain estimates on $\tau_- (t_0, x_0, h)$ and $X(\tau_- (t_0, x_0, h); t_0, x_0, h)$.

Lemma 5.2. *Let us assume $(t_0, x_0, h) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$. Then, if*

$$(5.18) \quad b \in L_{x_d}^-(S_{t_1} \times \mathbb{R}^p) \quad (\text{see (2.19)}),$$

then there exists $\delta > 0$ such that, under the conditions

$$(\bar{t}_0, \bar{x}_0, \bar{h}) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p,$$

$$|x_0 - \bar{x}_0| \leq \delta, \quad |x_0 - \bar{x}_0| \leq \delta, \quad \|h - \bar{h}\|_\infty \leq \delta,$$

we have

$$(5.19) \quad \begin{aligned} & |\tau_- (t_0, x_0, h) - \tau_- (\bar{t}_0, \bar{x}_0, \bar{h})| + \\ & + |X(\tau_- (t_0, x_0, h); t_0, x_0, h) - X(\tau_- (\bar{t}_0, \bar{x}_0, \bar{h}); \bar{t}_0, \bar{x}_0, \bar{h})| \leq \\ & \leq C_3 [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty], \end{aligned}$$

where

$$(5.20) \quad \begin{aligned} C_3 = & C \left(1 + \|b\|_{L^\infty(0,t_1;h,\bar{h})}^2 + \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})}^2 \right) \times \\ & \times \exp \left[C \left(\|b\|_{L^\infty(0,t_1;h,\bar{h})} + \Lambda(D_x h, D_x \bar{h}) \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})} \right) \right] \end{aligned}$$

with a constant C depending only on B_d , $\|D_{(t,x')}b_d\|_{L_{x_d}^1(0,1;L_{(t,x')}^\infty(S'_{t_1}))}$.

Furthermore, for every $s \in (\tau_- (t_0, x_0, h), t_0)$, there exists $\delta > 0$ such that, under the conditions

$$\begin{aligned} & (\bar{t}_0, \bar{x}_0, \bar{h}) \in S_{t_1} \times W^{1,\infty}(S_{t_1})^p, \quad \bar{s} \in (\tau_- (\bar{t}_0, \bar{x}_0, \bar{h}), \bar{t}_0), \\ & |t_0 - \bar{t}_0| \leq \delta, \quad |x_0 - \bar{x}_0| \leq \delta, \quad \|h - \bar{h}\|_\infty \leq \delta, \quad |s - \bar{s}| \leq \delta, \end{aligned}$$

we have

$$(5.21) \quad |X(s; t_0, x_0, h) - X(\bar{s}; \bar{t}_0, \bar{x}_0, \bar{h})| \leq C_4 [|t_0 - \bar{t}_0| + |s - \bar{s}| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty],$$

where

$$(5.22) \quad \begin{aligned} C_4 = & C \left[1 + \|b\|_{L^\infty(0,t_1;h,\bar{h})} + \left(1 \vee \|b\|_{L^\infty(0,t_1;h,\bar{h})} \vee \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})} \right) \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})} \times \right. \\ & \left. \times \Lambda(D_x h, D_x \bar{h}) \right] \exp \left(\Lambda(D_x h, D_x \bar{h}) \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})} \right), \end{aligned}$$

(C is a positive constant not depending on b, h , etc.).

Hence, $\tau_- \in W_{(t,x,y_{loc})}^{1,\infty}(S_{t_1} \times \mathbb{R}^p)$, $X \circ \tau_- \in W_{(t,x,y_{loc})}^{1,\infty}(S_{t_1} \times \mathbb{R}^p)^d$.

Proof. Now, assuming (5.18), we observe that $X_d(\cdot; t_0, x_0, h) : [\tau_- (t_0, x_0, h), \tau_+ (t_0, x_0, h)] \rightarrow [X_d(\tau_+ (t_0, x_0, h); t_0, x_0, h), X_d(\tau_- (t_0, x_0, h); t_0, x_0, h)]$ is absolute continuous and strictly decreasing; therefore, the inverse function of X_d ,

which we denote by $T(\cdot; t_0, x_0, h)$, is continuous and strictly decreasing. Hence, it is possible to apply the chain rule to the following function

$$(5.23) \quad id|_{[X_d(\tau_+(t_0, x_0, h); t_0, x_0, h), X_d(\tau_-(t_0, x_0, h); t_0, x_0, h)]} = X_d \circ T,$$

(see [25]). Therefore, we obtain :

$$(5.24) \quad \partial_{x_d} \Phi(x_d) = \frac{1}{b_d(\Phi(x_d), X'(\Phi(x_d); t_0, x_0, h), x_d)}$$

for x_d -almost everywhere on $[X_d(\tau_+(t_0, x_0, h); t_0, x_0, h), X_d(\tau_-(t_0, x_0, h); t_0, x_0, h)]$, where

$$(5.25) \quad \Phi(\cdot) = T(\cdot; t_0, x_0, h).$$

Hence, we can associate to (5.24) the following integral equation

$$(5.26) \quad \Phi(x_d) = t_0 + \int_{x_{0d}}^{x_d} \frac{dw}{b_d(\Phi(w), X'(\Phi(w); t_0, x_0, h), w)}.$$

for every $x_d \in [X_d(\tau_+(t_0, x_0, h); t_0, x_0, h), X_d(\tau_-(t_0, x_0, h); t_0, x_0, h)]$.

Afterwards, studying the equation (5.26) we deduce a key estimate to prove this lemma. Indeed, assuming $(\bar{t}_0, \bar{x}_0, \bar{h}) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$, the condition (5.18) and that exists β such that $x_{0d} \vee \bar{x}_{0d} \leq \beta \leq X_d(\tau_-(t_0, x_0, h); t_0, x_0, h) \wedge X_d(\tau_-(\bar{t}_0, \bar{x}_0, \bar{h}); \bar{t}_0, \bar{x}_0, \bar{h})$, then, after having defined $\bar{\Phi}(\cdot) = T(\cdot; \bar{t}_0, \bar{x}_0, \bar{h})$ and assumed $x_d \in [x_{0d} \vee \bar{x}_{0d}, \beta]$, we obtain

$$(5.27) \quad \begin{aligned} & |\Phi(x_d) - \bar{\Phi}(x_d)| \leq \\ & \leq |t_0 - \bar{t}_0| + B_d^{-1} |x_{0d} - \bar{x}_{0d}| + B_d^{-2} \int_{x_{0d} \vee \bar{x}_{0d}}^{x_d} \|\nabla_{(t, x')} b_d(\cdot, w)\|_{L^\infty(S'_{t_1})} \times \\ & \times [|\Phi(w) - \bar{\Phi}(w)| + |X'(\Phi(w); t_0, x_0, h) - X'(\bar{\Phi}(w); \bar{t}_0, \bar{x}_0, \bar{h})|] dw. \end{aligned}$$

Now, for every $M > 0$ there exists $\bar{\rho} > 0$ such that if $|x_0 - \bar{x}_0| \leq M$, $\|h - \bar{h}\|_\infty \leq M$ and $|t_0 - \bar{t}_0| \leq \bar{\rho}$, then we have

$$(5.28) \quad \tau_-(t_0, x_0, h) \vee \tau_-(\bar{t}_0, \bar{x}_0, \bar{h}) \leq t_0 \wedge \bar{t}_0.$$

This result can be obtained by (5.4). Therefore, assuming $\bar{t}_0 \leq t_0$ and $x_d \in [x_{0d} \vee \bar{x}_{0d}, \beta]$, we deduce the following possibilities

$$1) \tau_-(t_0, x_0, h) \leq \tau_-(\bar{t}_0, \bar{x}_0, \bar{h}).$$

Hence, thanks to (5.10) and (5.11), we obtain

$$(5.29) \quad \begin{aligned} & |X'(\Phi(w); t_0, x_0, h) - X'(\bar{\Phi}(w); t_0, \bar{x}_0, \bar{h})| \leq \\ & \leq |X(\Phi(w); t_0, x_0, h) - X(\bar{\Phi}(w); t_0, x_0, h)| + |X(\bar{\Phi}(w); t_0, x_0, h) - X(\bar{\Phi}(w); \bar{t}_0, \bar{x}_0, \bar{h})| \leq \\ & \leq \|b\|_{L^\infty(0, t_1; h)} |\Phi(w) - \bar{\Phi}(w)| + C_2 [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty], \end{aligned}$$

for all $w \in [x_{0d} \vee \bar{x}_{0d}, x_d]$.

$$2) \tau_-(\bar{t}_0, \bar{x}_0, \bar{h}) \leq \tau_-(t_0, x_0, h).$$

It is not restrictive to assume $\bar{\Phi}(w) \leq \tau_-(t_0, x_0, h)$ and $\bar{t}_0 \leq \Phi(w)$ with $w \in [x_{0d} \vee \bar{x}_{0d}, x_d]$. therefore, thanks to (5.10) and (5.11), we have the following estimate

$$\begin{aligned}
(5.30) \quad & |X'(\Phi(w); t_0, x_0, h) - X'(\bar{\Phi}(w); t_0, \bar{x}_0, \bar{h})| \leq \\
& \leq |X(\Phi(w); t_0, x_0, h) - X(\bar{t}_0; t_0, x_0, h)| + |X(\bar{t}_0; t_0, x_0, h) - X(\bar{t}_0; \bar{t}_0, \bar{x}_0, \bar{h})| \leq \\
& + |X(\bar{t}_0; t_0, x_0, h) - X(\bar{\Phi}(w); \bar{t}_0, \bar{x}_0, \bar{h})| \leq \|b\|_{L^\infty(0, t_1; h)} (\Phi(w) - \bar{t}_0) + \\
& + C_2 [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty] + \|b\|_{L^\infty(0, t_1; \bar{h})} (\bar{t}_0 - \bar{\Phi}(w)) \leq \\
& \leq \|b\|_{L^\infty(0, t_1; h, \bar{h})} |\Phi(w) - \bar{\Phi}(w)| + C_2 [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty],
\end{aligned}$$

for all $w \in [x_{0d} \vee \bar{x}_{0d}, x_d]$.

In a similar way, we obtain the estimate (5.30) for $t_0 \leq \bar{t}_0$. Hence, taking into account (5.30) and applying Gronwall's lemma to (5.27), we deduce the following estimates

$$(5.31) \quad |\Phi(x_d) - \bar{\Phi}(x_d)| \leq C_{3a} [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty],$$

$$\begin{aligned}
(5.32) \quad & |X'(\Phi(x_d); t_0, x_0, h) - X'(\bar{\Phi}(x_d); t_0, \bar{x}_0, \bar{h})| \leq C_{3b} [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty],
\end{aligned}$$

for all $x_d \in [x_{0d} \vee \bar{x}_{0d}, \beta]$, where C_{3a} and C_{3b} are so defined

$$\begin{aligned}
(5.33) \quad & C_{3a} = C \left(1 + \|D_{(x,y)}b\|_{L^1(0, t_1; h, \bar{h})} \right) \times \\
& \times \exp \left[C \left(\|b\|_{L^\infty(0, t_1; h, \bar{h})} + \Lambda(D_x h, D_x \bar{h}) \|D_{(x,y)}b\|_{L^1(0, t_1; h, \bar{h})} \right) \right],
\end{aligned}$$

$$\begin{aligned}
(5.34) \quad & C_{3b} = C \left(1 + \|b\|_{L^\infty(0, t_1; h, \bar{h})}^2 + \|D_{(x,y)}b\|_{L^1(0, t_1; h, \bar{h})}^2 \right) \times \\
& \times \exp \left[C \left(\|b\|_{L^\infty(0, t_1; h, \bar{h})} + \Lambda(D_x h, D_x \bar{h}) \|D_{(x,y)}b\|_{L^1(0, t_1; h, \bar{h})} \right) \right]
\end{aligned}$$

and C depends only on $B_d, \|D_{(t,x')}b_d\|_{L_{x_d}^1(0, 1; L_{(t,x')}^\infty(S'_{t_1}))}$.

After having obtained (5.7), (5.31) and (5.32), we can prove the following properties

A) if $\tau_-(t_0, x_0, h) = 0$, $X_d(\tau_-(t_0, x_0, h); t_0, x_0, h) < 1$ then there exists $\delta > 0$ such that $\tau_-(\bar{t}_0, \bar{x}_0, \bar{h}) = 0$ for $|t_0 - \bar{t}_0|, |x_0 - \bar{x}_0|, \|h - \bar{h}\|_{L^\infty(0, t_1; W^{1, \infty}(G))^p} < \delta$;

B) if $\tau_-(t_0, x_0, h) > 0$ there exists $\delta > 0$ such that $X_d(\tau_-(\bar{t}_0, \bar{x}_0, \bar{h}); \bar{t}_0, \bar{x}_0, \bar{h}) = 1$ for $|t_0 - \bar{t}_0|, |x_0 - \bar{x}_0|, \|h - \bar{h}\|_{L^\infty(0, t_1; W^{1, \infty}(G))^p} < \delta$.

We prove, for example, A). If A) is not true, then there exists a sequence $\{(t_{0n}, x_{0n}, h_n) | n \in \mathbb{N}\} \subseteq S_{t_1} \times L^\infty(0, t_1; W^{1, \infty}(G))^p$ such that

$$(5.35) \quad |t_0 - t_{0n}|, |x_0 - x_{0n}|, \|h - h_n\|_{L^\infty(0, t_1; W^{1, \infty}(G))^p} < \frac{1}{n+1},$$

$$\tau_-(t_{0n}, x_{0n}, h_n) > 0, \quad X_d(\tau_-(t_{0n}, x_{0n}, h_n); t_{0n}, x_{0n}, h_n) = 1 \quad \forall n \in \mathbb{N}.$$

We assume, with reference to (5.31), $\bar{t}_0 = t_{0n}$, $\bar{x}_0 = x_{0n}$ and $\bar{h} = h_n$; therefore, for n big enough, it is possible to take $\beta = X_d(\tau_-(t_0, x_0, h); t_0, x_0, h)$. Hence, assuming $x_d = \beta$ and applying (5.31), we deduce that for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$(5.36) \quad |\tau_-(t_0, x_0, h) - T(X_d(\tau_-(t_0, x_0, h); t_0, x_0, h); t_{0n}, x_{0n}, h_n)| = \\ = T(X_d(0; t_0, x_0, h); t_{0n}, x_{0n}, h_n) \leq \epsilon \quad \forall n \geq n_0,$$

therefore we deduce

$$(5.37) \quad \tau_-(t_{0n}, x_{0n}, h_n) \leq \epsilon \quad \forall n \geq n_0,$$

from which it follows immediately

$$(5.38) \quad \lim_{n \rightarrow \infty} \tau_-(t_{0n}, x_{0n}, h_n) = 0.$$

Applying (5.7), with $s = \tau_-(t_{0n}, x_{0n}, h_n)$, $\bar{t}_0 = t_{0n}$, $\bar{x}_0 = x_{0n}$, $\bar{h} = h_n$ we deduce that exists $n'_0 \in \mathbb{N}$, greater than n_0 , such that

$$(5.39) \quad |X_d(\tau_-(t_{0n}, x_{0n}, h_n); t_0, x_0, h) - X_d(\tau_-(t_{0n}, x_{0n}, h_n); t_{0n}, x_{0n}, h_n)| = \\ = |X_d(\tau_-(t_{0n}, x_{0n}, h_n); t_0, x_0, h) - 1| < \epsilon \quad \forall n \geq n'_0.$$

Hence, thanks to (5.38), (5.39) and to the continuity of $X_d(\cdot; t_0, x_0, h)$, we obtain $X_d(\tau_-(t_0, x_0, h); t_0, x_0, h) = 1$ and this is a contradiction; therefore, we have showed A).

Now, if the hypothesis of A) are verified, then we have

$$(5.40) \quad |\tau_-(t_0, x_0, h) - \tau_-(\bar{t}_0, \bar{x}_0, \bar{h})| = 0$$

for $|t_0 - \bar{t}_0|, |x_0 - \bar{x}_0|, \|h - \bar{h}\|_{L^\infty(0, t_1; W^{1, \infty}(G))^p} < \delta$. Therefore, thanks to (5.11), we deduce

$$(5.41) \quad |X(\tau_-(t_0, x_0, h); t_0, x_0, h) - X(\tau_-(\bar{t}_0, \bar{x}_0, \bar{h}); \bar{t}_0, \bar{x}_0, \bar{h})| \leq \\ C_2 [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty]$$

for $|t_0 - \bar{t}_0|, |x_0 - \bar{x}_0|, \|h - \bar{h}\|_{L^\infty(0, t_1; W^{1, \infty}(G))^p} < \delta$.

Instead, if the hypothesis of B) is verified, then, thanks to (5.31) and (5.32), with $\beta = 1$, we obtain

$$(5.42) \quad |\tau_-(t_0, x_0, h) - \tau_-(\bar{t}_0, \bar{x}_0, \bar{h})| \leq C_{3a} [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_{L^\infty(S_{t_1})}],$$

$$(5.43) \quad |X'(\tau_-(t_0, x_0, h); t_0, x_0, h) - X'(\tau_-(\bar{t}_0, \bar{x}_0, \bar{h}); \bar{t}_0, \bar{x}_0, \bar{h})| \leq \\ C_{3b} [|t_0 - \bar{t}_0| + |x_0 - \bar{x}_0| + \|h - \bar{h}\|_\infty]$$

for $|t_0 - \bar{t}_0|, |x_0 - \bar{x}_0|, \|h - \bar{h}\|_{L^\infty(0, t_1; W^{1, \infty}(G))^p} < \delta$.

Of course it is possible that A) and B) are not verified; in this case, we deduce that $\tau_-(t_0, x_0, h) = 0$ and $X_d(\tau_-(t_0, x_0, h); t_0, x_0, h) = 1$, therefore we can apply (5.40)-(5.41) or (5.42)-(5.43).

Hence, we have showed (5.19).

Now, after having fixed $s \in (\tau_-(t_0, x_0, h), t_0)$ and using the regularity of $\tau_-(\cdot)$, it is possible to find $\delta > 0$ such that

$$(5.44) \quad |X(s; t_0, x_0, h) - X(\bar{s}; \bar{t}_0, \bar{x}_0, \bar{h})| \leq |x_0 - \bar{x}_0| + \int_{I(s, \bar{s}) \cup I(t_0, \bar{t}_0)} \|b(r, \cdot)\|_{L^\infty(G; h, \bar{h})} dr + \Delta$$

if $|t_0 - \bar{t}_0|, |x_0 - \bar{x}_0|, \|h - \bar{h}\|_{W^{1, \infty}(S_{t_1})^p}, |s - \bar{s}| \leq \delta$ and $\bar{s} \in (\tau_-(\bar{t}_0, \bar{x}_0, \bar{h}), \bar{t}_0)$, where

$$(5.45) \quad \Delta = \int_{s \vee \bar{s}}^{t_0 \wedge \bar{t}_0} |b(r, X(r; t_0, x_0, h), h(r, X(r; t_0, x_0, h))) - b(r, X(r; \bar{t}_0, \bar{x}_0, \bar{h}), \bar{h}(r, X(r; \bar{t}_0, \bar{x}_0, \bar{h})))| dr.$$

After simple calculations, we have

$$(5.46) \quad \Delta \leq \int_{s \vee \bar{s}}^{t_0 \wedge \bar{t}_0} \|D_{(x, y)} b\|_{L^\infty(G; h, \bar{h})} [\Lambda(D_x h, D_x \bar{h}) \times \\ \times |X(r; t_0, x_0, h) - X(r; \bar{t}_0, \bar{x}_0, \bar{h})| + \|h - \bar{h}\|_\infty] dr.$$

Therefore, we can obtain (5.21) by (5.44), (5.46) and (5.7).

□

Now we are ready to study the continuous dependence of flow X and initial time τ_- on vector field b and parametric vector function h .

Lemma 5.3. *If $b^{(k)} \in L_t^1(0, t_1; W_{(x, y)_{loc}}^{1, \infty}(G \times \mathbb{R}^p))^d$, $B_d^{(k)} > 0$ such that $b_d^{(k)}(t, x) \leq -B_d^{(k)} \forall x \in G$ a.e. $t \in (0, t_1)$, $(t_0, x_0, h^{(k)}) \in S_{t_1} \times L^\infty(0, T; W^{1, \infty}(G))^p$; $X^{(k)}(\cdot; t_0, x_0, h^{(k)}) : [\tau_-^{(k)}(t_0, x_0, h^{(k)}), t_0] \rightarrow \mathbb{R}^d$ is the flux associated with $b^{(k)}$ where $k = 1, 2$, then we have*

$$(5.47) \quad |X^{(2)}(s; t_0, x_0, h^{(2)}) - X^{(1)}(s; t_0, x_0, h^{(1)})| \leq \\ \leq C_1 \left(\|h^{(2)} - h^{(1)}\|_\infty + \|b^{(2)} - b^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} \right),$$

for every $s \in [\tau_-^{(1)}(t_0, x_0, h^{(1)}) \vee \tau_-^{(2)}(t_0, x_0, h^{(2)}), t_0]$; furthermore, the constant C_1 is determined by assuming $b = b^{(1)}$, $h = h^{(1)}$ and $\bar{h} = h^{(2)}$.

Finally, if $b^{(k)} \in L_{x_d}^-(S_{t_1} \times \mathbb{R}^p)$ (see (2.19)) with $k = 1, 2$, then for every $M > 0$ there exists $\delta > 0$ depending on $h^{(1)}, b^{(1)}, M$, such that, under the conditions $\|h^{(2)} - h^{(1)}\|_\infty < \delta$, $\|b^{(2)} - b^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} < \delta$ and $\|b^{(k)}\|_{L^\infty(0, t_1; h^{(k)})} \leq M$ with $k = 1, 2$, we have

$$(5.48) \quad \left| \tau_-^{(2)}(t_0, x_0, h^{(2)}) - \tau_-^{(1)}(t_0, x_0, h^{(1)}) \right| + \\ + \left| X^{(2)}(\tau_-^{(2)}(t_0, x_0, h^{(2)}); t_0, x_0, h^{(2)}) - X^{(1)}(\tau_-^{(1)}(t_0, x_0, h^{(1)}); t_0, x_0, h^{(1)}) \right| \leq$$

$$\leq C_5 \left(\|h^{(2)} - h^{(1)}\|_\infty + \|b^{(2)} - b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right),$$

where

$$(5.49) \quad C_5 = C \left(1 + \bigvee_{k=1,2} \|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} \right) \left(1 + \|D_{(x,y)} b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right) \times \\ \times \exp \left[C \left(\bigvee_{k=1,2} \|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} + \Lambda(D_x h^{(1)}, D_x h^{(2)}) \|D_{(x,y)} b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right) \right]$$

with the constant C depending only on $B_d^{(k)}$, $\|b_d^{(k)}\|_{L^1_{x_d}(0,1;L^\infty_{(t,x')}(S'_{t_1}))}$ (with $k = 1, 2$), $\|D_{(t,x')} b_d^{(1)}\|_{L^1_{x_d}(0,1;L^\infty_{(t,x')}(S'_{t_1}))}$.

Proof.

Let us define

$$(5.50) \quad X^{(k)}(s; t_0, x_0; h^{(k)}) = x_0 - \int_s^{t_0} b^{(k)}(r, X^{(k)}(r; t_0, x_0; h^{(k)}); h^{(k)}(r, X^{(k)}(r; t_0, x_0; h^{(k)}))) dr,$$

where $s \in [\tau_-^{(k)}(t_0, x_0, h^{(k)}), t_0]$ and $k = 1, 2$. We obtain (5.47) by the difference between (5.50) with $k = 2$ and (5.50) with $k = 1$.

Now, using similar arguments to those employed to obtain the equation (5.26), we can transform the equation (5.50) in the following vectorial integral equation

$$(5.51) \quad \Phi^{(k)}(x_d) = t_0 + \int_{x_{0d}}^{x_d} \frac{dw}{b_d^{(k)}(\Phi^{(k)}(w), X'^{(k)}(\Phi^{(k)}(w); Z_0^{(k)}), w)},$$

$$(5.52) \quad X'^{(k)}(\Phi^{(k)}(x_d); Z_0^{(k)}) = x'_0 + \\ + \int_{x_{0d}}^{x_d} \frac{b'^{(k)}(\Phi^{(k)}(w), X'^{(k)}(\Phi^{(k)}(w); Z_0^{(k)}), w, h^{(k)}(\Phi^{(k)}(w), X'^{(k)}(\Phi^{(k)}(w); Z_0^{(k)}), w))}{b_d^{(k)}(\Phi^{(k)}(w), X'^{(k)}(\Phi^{(k)}(w); Z_0^{(k)}), w)} dw,$$

for every $x_d \in [x_{0d}, X_d^{(k)}(\tau_-^{(k)}(t_0, x_0, h^{(k)}); t_0, x_0, h^{(k)})]$, $k = 1, 2$, where we have defined

$$(5.53) \quad Z_0^{(k)} = (t_0, x_0, h^{(k)}), \quad \Phi^{(k)}(\cdot) = T^{(k)}(\cdot; Z_0^{(k)}),$$

and $T^{(k)}(\cdot; Z_0^{(k)})$ is the inverse function of $X^{(k)}(\cdot; Z_0^{(k)})$.

Studying the difference between (5.51) with $k = 2$ and (5.51) with $k = 1$, we can deduce the following estimate

$$(5.54) \quad |\Phi^{(2)}(x_d) - \Phi^{(1)}(x_d)| \leq B_d^{(2)-1} B_d^{(1)-1} \left[\|b_d^{(2)} - b_d^{(1)}\|_{L^1_{x_d}(0,1;h^{(1)},h^{(2)})} + \right. \\ \left. + \int_{x_{0d}}^{x_d} \|D_{(t,x')} b_d^{(1)}(\cdot, w)\|_{L^\infty(S'_{t_1})} (|\Phi^{(2)}(w) - \Phi^{(1)}(w)| + \right.$$

$$+|X'^{(2)}(\Phi^{(2)}(w); Z_0^{(2)}) - X'^{(1)}(\Phi^{(1)}(w); Z_0^{(1)})|dw],$$

where $x_d \in [x_{0d}, \bigwedge_{k=1,2} X_d^{(k)}(\tau_-^{(k)}(t_0, x_0, h^{(k)}); t_0, x_0, h^{(k)})]$.

Now, thanks to (5.10) and (5.47), we have

$$(5.55) \quad |X'^{(2)}(\Phi^{(2)}(w); Z_0^{(2)}) - X'^{(1)}(\Phi^{(1)}(w); Z_0^{(1)})| \leq \bigvee_{k=1,2} \|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} |\Phi^{(2)}(w) - \Phi^{(1)}(w)| + \\ + C_1 \left(\|h^{(2)} - h^{(1)}\|_\infty + \|b^{(2)} - b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right), \quad \forall w \in [x_{0d}, x_d].$$

Hence, applying (5.55) in (5.54) and Gronwall's lemma, we deduce

$$(5.56) \quad |T^{(2)}(x_d; t_0, x_0, h^{(2)}) - T^{(1)}(x_d; t_0, x_0, h^{(1)})| \leq \\ \leq C'_{3a} \left(\|h^{(2)} - h^{(1)}\|_\infty + \|b^{(2)} - b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right),$$

$$(5.57) \quad |X'^{(2)}(T^{(2)}(x_d; t_0, x_0, h^{(2)}); t_0, x_0, h^{(2)}) - X'^{(1)}(T^{(1)}(x_d; t_0, x_0, h^{(1)}); t_0, x_0, h^{(1)})| \leq \\ \leq C'_{3b} \left(\|h^{(2)} - h^{(1)}\|_\infty + \|b^{(2)} - b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right),$$

where

$$(5.58) \quad C'_{3a} = C \left(1 + \|D_{(x,y)} b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right) \times \\ \times \exp \left[C \left(\bigvee_{k=1,2} \|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} + \Lambda(D_x h^{(1)}, D_x h^{(2)}) \|D_{(x,y)} b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right) \right],$$

$$(5.59) \quad C'_{3b} = \left(1 + \bigvee_{k=1,2} \|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} \right) C'_{3a}.$$

After having obtained (5.47), (5.56), (5.57) and assuming $M > 0$, we can prove the following properties

A') if $\tau_-^{(1)}(t_0, x_0, h^{(1)}) = 0$, $X_d^{(1)}(\tau_-^{(1)}(t_0, x_0, h^{(1)}); t_0, x_0, h^{(1)}) < 1$ then there exists $\delta > 0$ such that $\tau_-^{(2)}(t_0, x_0, h^{(2)}) = 0$ for $\|h^{(1)} - h^{(2)}\|_\infty, \|b^{(1)} - b^{(2)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} < \delta$ and $\|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} \leq M$ with $k = 1, 2$;

B') if $\tau_-^{(1)}(t_0, x_0, h^{(1)}) > 0$ there exists $\delta > 0$ such that $X_d^{(2)}(\tau_-^{(2)}(t_0, x_0, h^{(2)}); t_0, x_0, h^{(2)}) = 1$ for $\|h^{(1)} - h^{(2)}\|_\infty, \|b^{(1)} - b^{(2)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} < \delta$ and $\|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} \leq M$ with $k = 1, 2$.

Using (5.47), (5.56) and (5.57), the proofs of A'), B') may be carried out as those used to show A) and B) (see the proof of Lemma 5.2). At this point, to show (5.48), we use the same argument employed in the proof of (5.19).

□

6 - The well-posedness of the IBVP for the linear transport equation.

Before studying the initial and boundary value problem (4.1)-(4.2), we give the definition of generalized solution for this problem following the one given in [27].

Definition 6.0. *Let us assume $h \in L^\infty(0, t_1; W^{1,\infty}(G))^p$, (5.1)-(5.2) for the vector field b and*

$$(6.1) \quad z^* \in C^0(\Gamma_-) \cap L^\infty(\Gamma_-),$$

$$(6.2) \quad c, a \in L_t^\infty(0, t_1; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p)) \cap L_t^1(0, t_1; W_{(x, y_{loc})}^{1,\infty}(G \times \mathbb{R}^p)).$$

Hence, we say that the function $z(\cdot, h) \in L^\infty(0, t_1; W^{1,\infty}(G))$ is a generalized solution for the IBVP (4.1)-(4.2), if and only if

$$(6.3) \quad \begin{aligned} z(t, x; h) &= z^*(\tau_-(t, x, h), X(\tau_-(t, x, h); t, x, h)) + \\ &+ \int_{\tau_-(t, x, h)}^t [-c(s, X(s; t, x, h), h(s, X(s; t, x, h))) z(s, X(s; t, x, h); h) + \\ &+ a(s, X(s; t, x, h), h(s, X(s; t, x, h)))] ds \end{aligned}$$

for every $(t, x) \in S_{t_1}$.

Now, we are ready to give some results on the existence, uniqueness, regularity and continuous dependence on data of the generalized solution for the problem (4.1)-(4.2).

Lemma 6.1. *Let us assume $b \in L_{x_d}^-(S_{t_1})$, $z^* \in Lip_{loc}^{unif}(\Gamma_-)$ and (6.2) for c, a . Furthermore, we denote by L^* a positive constant such that*

$$(6.4) \quad |z^*(t, x) - z^*(\bar{t}, \bar{x})| \leq L^* \{|t - \bar{t}| + |x - \bar{x}|\},$$

where $(t, x), (\bar{t}, \bar{x}) \in \Gamma_-$ and there exists a positive number $\sigma > 0$ such that $|t - \bar{t}| + |x - \bar{x}| \leq \sigma$.

Then, for every $h \in L^\infty(0, t_1; W^{1,\infty}(G))^p$ there exists one and only one generalized solution $z(\cdot, h) \in L^\infty(0, t_1; W^{1,\infty}(G))$ for IBVP (4.1)-(4.2).

Furthermore, this solution satisfies the following inequality

$$(6.5) \quad \|z(\cdot, h)\|_{L^\infty(S_{t_1})} \leq \left[\|z^*\|_{L^\infty(\Gamma_-)} + \|a\|_{L^1(0, t_1; h)} \right] \exp \|c\|_{L^1(0, t_1; h)}.$$

Proof.

First of all, we show the uniqueness of the generalized solution. Hence, we suppose that $z(\cdot, h)$ and $w(\cdot, h)$ are two generalized solutions for the IBVP (4.1)-(4.2). Therefore, we immediately deduce

$$(6.6) \quad \begin{aligned} &|z(t, x; h) - w(t, x; h)| \leq \\ &\leq \|c\|_{L^\infty(0, t_1; h)} \int_{\tau_-(t, x, h)}^t |z(s, X(\tau_-(t, x, h); t, x, h); h) - w(s, X(\tau_-(t, x, h); t, x, h))| ds. \end{aligned}$$

After simple calculations and applying Gronwall's lemma we obtain $z = w$.

Using Gronwall's lemma, the estimate (6.5) follows immediately.

Now, we prove the existence of the generalized solution ; for this purpose, we can use the method of successive approximations to prove the existence of generalized solution. Assuming $z_1(\cdot, h) \in L^\infty(0, t_1; W^{1,\infty}(G))$, we consider the following recurrence schema

$$(6.7) \quad \begin{aligned} z_{n+1}(t, x; h) &= z^*(\tau_-(t, x, h), X(\tau_-(t, x, h); t, x, h)) + \\ &+ \int_{\tau_-(t, x, h)}^t [-c(s, X(s; t, x, h), h(s, X(s; t, x, h))) z_n(s, X(s; t, x, h); h) + \\ &+ a(s, X(s; t, x, h), h(s, X(s; t, x, h)))] ds, \quad \forall n > 1. \end{aligned}$$

We must verify that $z_{n+1}(\cdot; h)$ is well defined and belongs to $L^\infty(0, t_1; W^{1,\infty}(G))$. To show this, it is sufficient to see that if we assume the following inductive hypothesis : $z_2(\cdot; h), \dots, z_n(\cdot; h) \in L^\infty(0, t_1; W^{1,\infty}(G))$, then $z_{n+1}(\cdot; h) \in L^\infty(0, t_1; W^{1,\infty}(G))$.

It is not so hard to obtain the following estimates :

$$(6.8) \quad \|z_{n+1}(\cdot, h) - z_n(\cdot, h)\|_{L^\infty(S_{t_1})} \leq \|c\|_{L^\infty(0,1;h)}^{n-1} \|z_2(\cdot, h) - z_1(\cdot, h)\|_{L^\infty(S_{t_1})} \frac{t_1^{n-1}}{(n-1)!},$$

$$(6.9) \quad \begin{aligned} \|z_{n+1}(\cdot; h)\|_{L^\infty(S_{t_1})} &\leq \|z^*\|_{L^\infty(\Gamma_-)} \sum_{k=0}^{n-1} \|c\|_{L^\infty(0,t_1;h)}^k \frac{t_1^k}{k!} + \|a\|_{L^\infty(0,t_1;h)} \sum_{k=0}^{n-1} \|c\|_{L^\infty(0,t_1;h)}^k \frac{t_1^{k+1}}{(k+1)!} + \\ &+ \|c\|_{L^\infty(0,t_1;h)}^n \|z_1(\cdot; h)\|_{L^\infty(S_{t_1})} \frac{t_1^n}{n!}. \end{aligned}$$

After these preliminary estimates, we are ready to study the regularity of $z_{n+1}(\cdot; h)$. Assuming $(t, x), (t, \bar{x}) \in S_{t_1}$ and remembering (6.7), we deduce

$$(6.10) \quad |z_{n+1}(t, x; h) - z_{n+1}(t, \bar{x}; h)| \leq \Delta^* + \Delta_a + \Delta_c.$$

$\Delta^*, \Delta_a, \Delta_c$ are so defined

$$(6.11) \quad \begin{aligned} \Delta^* &= |z^*(\tau_-(t, x, h), X(\tau_-(t, x, h); t, x, h)) - z^*(\tau_-(t, \bar{x}, h), X(\tau_-(t, \bar{x}, h); t, \bar{x}, h))| \leq \\ &\leq L^* (|\tau_-(t, x, h) - \tau_-(t, \bar{x}, h)| + |X(\tau_-(t, x, h); t, x, h) - X(\tau_-(t, \bar{x}, h); t, \bar{x}, h)|) \leq \\ &\leq L^* C_3 |x - \bar{x}|, \end{aligned}$$

where the last inequality can be deduced applying (5.19) and assuming $x_0 = x, \bar{x}_0 = \bar{x}, \bar{h} = h$ and $|x - \bar{x}| \leq \delta(t, x, h)$;

$$(6.12) \quad \begin{aligned} \Delta_a &= \left| \int_{\tau_-(t, x, h)}^t a(s, X(s; t, x, h), h(s; X(s; t, x, h))) ds - \right. \\ &\left. - \int_{\tau_-(t, \bar{x}, h)}^t a(s, X(s; t, \bar{x}, h), h(s; X(s; t, \bar{x}, h))) ds \right| \leq \int_{I(\tau_-(t, x, h), \tau_-(t, \bar{x}, h))} \|a(s, \cdot)\|_{L^\infty(G; h)} ds + \end{aligned}$$

$$\begin{aligned}
& + \int_{\tau_-(t,x,h) \vee \tau_-(t,\bar{x},h)}^t \left\| \nabla_{(x,y)} a(s, \cdot) \right\|_{L^\infty(G;h)} \left[|X(s; t, x, h) - X(s; t, \bar{x}, h)| + \right. \\
& \quad \left. + |h(s; X(s; t, x, h)) - h(s; X(s; t, \bar{x}, h))| \right] ds \leq \\
& \leq \left(\Lambda(D_x h, D_x h) C_2 \left\| \nabla_{(x,y)} a \right\|_{L^1(0,t_1;h)} + C_3 \|a\|_{L^\infty(0,t_1;h)} \right) |x - \bar{x}|,
\end{aligned}$$

where the last inequality has been obtained applying (5.11), (5.19) and assuming $|x - \bar{x}| \leq \delta(t, x, h)$;

$$\begin{aligned}
(6.13) \quad \Delta_c &= \left| \int_{\tau_-(t,x,h)}^t c(s, X(s; t, x, h), h(s; X(s; t, x, h))) z_n(s, X(s; t, x, h); h) ds - \right. \\
& \quad \left. - \int_{\tau_-(t,\bar{x},h)}^t c(s, X(s; t, \bar{x}, h), h(s; X(s; t, \bar{x}, h))) z_n(s, X(s; t, \bar{x}, h); h) ds \right| \leq \\
& \leq \|z_n(\cdot; h)\|_{L^\infty(S_{t_1})} \left(\Lambda(D_x h, D_x h) C_2 \left\| \nabla_{(x,y)} c \right\|_{L^1(0,t_1;h)} + C_3 \|c\|_{L^\infty(0,t_1;h)} \right) |x - \bar{x}| + \\
& \quad + \|c\|_{L^\infty(0,t_1;h)} \int_0^t |z_n(s, x; h) - z_n(s, \bar{x}; h)| ds, \quad \text{where } |x - \bar{x}| \leq \delta(t, x, h).
\end{aligned}$$

Therefore, taking into account (6.9)-(6.13), we deduce there exists a constant $C > 0$ such that

$$(6.14) \quad |z_{n+1}(t, x; h) - z_{n+1}(t, \bar{x}; h)| \leq C |x - \bar{x}| + \|c\|_{L^\infty(0,t_1;h)} \int_0^t |z_n(s, x; h) - z_n(s, \bar{x}; h)| ds,$$

where $|x - \bar{x}| \leq \delta(t, x, h)$. Hence $z_{n+1}(\cdot, h) \in L^\infty(0, t_1; W^{1,\infty}(G))$ and, moreover, thanks to (6.7), is defined on all S_{t_1} .

Now, using (6.8), we observe that the telescopic series $z_1(\cdot, h) + \sum_{n=1}^{+\infty} (z_{n+1}(\cdot, h) - z_n(\cdot, h))$ is uniform convergent on S_{t_1} to a function that we denote by $z(\cdot, h)$. Finally, it is not so hard to verify, taking the limit for $n \rightarrow \infty$ in (6.14) and (6.7), that $z(\cdot, h) \in L^\infty(0, t_1; W^{1,\infty}(G))$ and it is the generalized solution for the IBVP (4.1)-(4.2).

□

Lemma 6.2. *Let us assume the hyphoteses of Lemma 6.1.*

For every $(t, x, h) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$ there exists $\delta > 0$ such that, under the conditions

$$(6.15) \quad |x - \bar{x}| \leq \delta, \quad \|h - \bar{h}\|_\infty \leq \delta, \quad (\bar{x}, \bar{h}) \in G \times L^\infty(0, t_1; W^{1,\infty}(G))^p,$$

we have

$$(6.16) \quad |z(t, x; h) - z(t, \bar{x}; \bar{h})| \leq C_6 (|x - \bar{x}| + \|h - \bar{h}\|_\infty),$$

where

$$\begin{aligned}
(6.17) \quad C_6 = & C \left[1 + \|b\|_{L^\infty(0,t_1;h,\bar{h})}^2 + \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})}^2 \right] \times \\
& \times \left[1 + L^{*2} + \|z^*\|_{L^\infty(\Gamma_-)}^2 + \|c\|_{L^\infty(0,t_1;h,\bar{h})}^2 + \|a\|_{L^\infty(0,t_1;h,\bar{h})}^2 + \right. \\
& \left. + \Lambda (D_x h, D_x \bar{h})^2 (\|\nabla_{(x,y)}c\|_{L^1(0,t_1;h,\bar{h})}^2 + \|\nabla_{(x,y)}a\|_{L^1(0,t_1;h,\bar{h})}^2) \right] \times \\
& \times \exp \left\{ C \left[\|b\|_{L^\infty(0,t_1;h,\bar{h})} + \Lambda (D_x h, D_x \bar{h}) \|D_{(x,y)}b\|_{L^1(0,t_1;h,\bar{h})} + \|c\|_{L^\infty(0,t_1;h,\bar{h})} \right] \right\}
\end{aligned}$$

and C is a constant depending on B_d , $\|D_{(t,x')}b_d\|_{L^1_d(0,1;L^\infty_{(t,x')}(S'_{t_1}))}$.

Hence, $z \in L^\infty_t(0, t_1; W^{1,\infty}_{(x,y_{loc})}(G \times \mathbb{R}^p))$.

Proof.

To prove (6.16), it is necessary to obtain preliminary estimates. We first deduce

$$(6.18) \quad |z(t, x; h) - z(t, \bar{x}; \bar{h})| \leq \tilde{\Delta}^* + \tilde{\Delta}_a + \tilde{\Delta}_c,$$

where

$$(6.19) \quad \tilde{\Delta}^* = |z^*(\tau_-(t, x, h), X(\tau_-(t, x, h); t, x, h)) - z^*(\tau_-(t, \bar{x}, \bar{h}), X(\tau_-(t, \bar{x}, \bar{h}); t, \bar{x}, \bar{h}))|,$$

$$\begin{aligned}
(6.20) \quad \tilde{\Delta}_a = & \left| \int_{\tau_-(t,x,h)}^t a(s, X(s; t, x, h), h(s, X(s; t, x, h))) ds - \right. \\
& \left. + \int_{\tau_-(t,\bar{x},\bar{h})}^t a(s, X(s; t, \bar{x}, \bar{h}), \bar{h}(s, X(s; t, \bar{x}, \bar{h}))) ds \right|
\end{aligned}$$

and

$$\begin{aligned}
(6.21) \quad \tilde{\Delta}_c = & \left| \int_{\tau_-(t,x,h)}^t c(s, X(s; t, x, h), h(s, X(s; t, x, h))) z(s, X(s; t, x, h); h) ds - \right. \\
& \left. + \int_{\tau_-(t,\bar{x},\bar{h})}^t c(s, X(s; t, \bar{x}, \bar{h}), \bar{h}(s, X(s; t, \bar{x}, \bar{h}))) z(s, X(s; t, \bar{x}, \bar{h}); \bar{h}) ds \right|.
\end{aligned}$$

Proceeding as for the estimates of Δ^* , Δ_a and Δ_c (see Lemma 6.1), we deduce there exist a positive constant \tilde{C} independent from t, x, h and $\delta(t, x, h) > 0$ such that if (6.25) is verified then we have

$$(6.22) \quad \tilde{\Delta}^* \leq L^* C_3 (|x - \bar{x}| + \|h - \bar{h}\|_\infty),$$

$$(6.23) \quad \tilde{\Delta}_a \leq \tilde{C} \left(\Lambda (D_x h, D_x \bar{h}) C_2 \|\nabla_{(x,y)}a\|_{L^1(0,t_1;h,\bar{h})} + C_3 \|a\|_{L^\infty(0,t_1;h,\bar{h})} \right) (|x - \bar{x}| + \|h - \bar{h}\|_\infty),$$

$$\begin{aligned}
(6.24) \quad \tilde{\Delta}_c &\leq \tilde{C} \left[\|z^*\|_{L^\infty(\Gamma_-)} + \|a\|_{L^1(0,t_1;h,\bar{h})} \right] \exp \|c\|_{L^1(0,t_1;h,\bar{h})} \times \\
&\left(\Lambda(D_x h, D_x h) C_2 \|\nabla_{(x,y)} c\|_{L^1(0,t_1;h,\bar{h})} + C_3 \|c\|_{L^\infty(0,t_1;h,\bar{h})} \right) (|x - \bar{x}| + \|h - \bar{h}\|_\infty) + \\
&+ \int_{\tau_-(t,x,h) \vee \tau_-(t,\bar{x},\bar{h})}^t \|c(s, \cdot)\|_{L^\infty(G;h,\bar{h})} |z(s, X(s; t, x, h); h) - z(s, X(s; t, \bar{x}, \bar{h}); \bar{h})| ds,
\end{aligned}$$

Then, thanks to (6.22), (6.23) and (6.24) we obtain an estimate for $|z(t, x; h) - z(t, \bar{x}; \bar{h})|$ (see (6.18)); afterwards, applying Gronwall's lemma we deduce (6.16). \square

It is important to observe that, assuming the same hypotheses of Lemma 6.1, we can deduce a stronger regularity result for z respect to the one obtained by Lemma 6.2. Indeed, we have

Corollary 6.3. *Let us assume the hypotheses of Lemma 6.1.*

For every $(t, x, h) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p$ there exists $\delta > 0$ such that, under the conditions

$$(6.25) \quad |t - \bar{t}| \leq \delta, \quad |x - \bar{x}| \leq \delta, \quad \|h - \bar{h}\|_\infty \leq \delta, \quad (\bar{t}, \bar{x}, \bar{h}) \in S_{t_1} \times L^\infty(0, t_1; W^{1,\infty}(G))^p,$$

we have

$$(6.26) \quad |z(t, x; h) - z(\bar{t}, \bar{x}; \bar{h})| \leq C_7 (|t - \bar{t}| + |x - \bar{x}| + \|h - \bar{h}\|_\infty),$$

where

$$\begin{aligned}
(6.27) \quad C_7 &= C \left[1 + L^* \right] \left[1 + \|b\|_{L^\infty(0,t_1;h,\bar{h})}^4 + \|D_{(x,y)} b\|_{L^1(0,t_1;h,\bar{h})}^4 \right] \times \\
&\times \left[1 + L^{*2} + \|z^*\|_{L^\infty(\Gamma_-)}^2 + \|c\|_{L^\infty(0,t_1;h,\bar{h})}^2 + \|a\|_{L^\infty(0,t_1;h,\bar{h})}^2 + \right. \\
&+ \Lambda(D_x h, D_x \bar{h})^2 (\|\nabla_{(x,y)} c\|_{L^1(0,t_1;h,\bar{h})}^2 + \|\nabla_{(x,y)} a\|_{L^1(0,t_1;h,\bar{h})}^2) \left. \right] \times \\
&\times \exp \left\{ C \left[\|b\|_{L^\infty(0,t_1;h,\bar{h})} + \Lambda(D_x h, D_x \bar{h}) \|D_{(x,y)} b\|_{L^1(0,t_1;h,\bar{h})} + \|c\|_{L^\infty(0,t_1;h,\bar{h})} \right] \right\}
\end{aligned}$$

and C is a constant depending on $t_1, B_d, \|D_{(t,x')} b_d\|_{L_{x_d}^1(0,1;L_{(t,x')}^\infty(S'_{t_1}))}$.

Therefore, we deduce that $z \in W_{(t,x,y_{loc})}^{1,\infty}(S_{t_1} \times \mathbb{R}^p)$.

Proof.

To show (6.26), we observe that there exists a positive number δ such that if $|t - \bar{t}| \leq \delta$, $|x - \bar{x}| \leq \delta$ and $\|h - \bar{h}\|_\infty \leq \delta$, then it follows the estimate

$$\begin{aligned}
(6.28) \quad &|z(t, x; h) - z(\bar{t}, \bar{x}; \bar{h})| \leq \\
&\leq |z^*(\tau_-(t, x, h), X(\tau_-(t, x, h); t, x, h)) - z^*(\tau_-(\bar{t}, \bar{x}, \bar{h}), X(\tau_-(\bar{t}, \bar{x}, \bar{h}); \bar{t}, \bar{x}, \bar{h}))| + \\
&\int_{I(\bar{t}, \bar{t}) \cup I(\tau_-(t, x, h), \tau_-(\bar{t}, \bar{x}, \bar{h}))} \left[\|c\|_{L^\infty(0,t_1;h,\bar{h})} (\|z(\cdot; h)\|_{L^\infty(S_{t_1})} \vee \|z(\cdot; \bar{h})\|_{L^\infty(S_{t_1})}) + \|a\|_{L^\infty(0,t_1;h,\bar{h})} \right] ds
\end{aligned}$$

$$\begin{aligned}
& + \int_{\tau_-(t,x,h) \vee \tau_-(\bar{t},\bar{x},\bar{h})}^{t \wedge \bar{t}} \left[\|z(\cdot; h)\|_{L^\infty(S_{t_1})} \|D_{(x,y)} c(s, \cdot)\|_{L^\infty(G; h, \bar{h})} + \|D_{(x,y)} a(s, \cdot)\|_{L^\infty(G; h, \bar{h})} \right] \times \\
& \quad \times \left[\Lambda(D_x h, D_x \bar{h}) |X(s; t, x, h) - X(s; \bar{t}, \bar{x}, \bar{h})| + \|h - \bar{h}\|_\infty \right] ds + \\
& + \int_{\tau_-(t,x,h) \vee \tau_-(\bar{t},\bar{x},\bar{h})}^{t \wedge \bar{t}} \|\bar{c}\|_{L^\infty(0, t_1; \bar{h})} |z(s, X(s; t, x, h); h) - z(s, X(s; \bar{t}, \bar{x}, \bar{h}); \bar{h})| ds.
\end{aligned}$$

Applying (5.11), (5.19) and (6.16) to (6.28), we deduce (6.26). \square

Remark 6.3.0. The estimate (6.16) is very useful to study the IBVP for our quasi-linear hyperbolic system (see the proof in Section 7). In particular, the key point is the expression of the constant C_6 , where the jacobian matrices of h and \bar{h} (parametric vector functions) are multiplied by norms in L^1 respect to time t_1 ; therefore these terms can be chosen small if t_1 is assumed sufficiently small.

We conclude this section with a result of continuous dependence of the generalized solution on the vector field b and the parameter function h .

Lemma 6.4. Assume $z^{*(k)} \in Lip_{loc}^{unif}(\Gamma_-)$, $c^{(k)}, a^{(k)} \in L_t^\infty(0, t_1; L_{(x, y_{loc})}^\infty(G \times \mathbb{R}^p)) \cap L_t^1(0, t_1; W_{(x, y_{loc})}^{1, \infty}(G \times \mathbb{R}^p))$ and let $(t, x, h^{(k)}, b^{(k)}) \in S_{t_1} \times L^\infty(0, t_1; W^{1, \infty}(G))^p \times L_{x_d}^-(S_{t_1} \times \mathbb{R}^p)$ with $k = 1, 2$. If $z^{(k)}$ is the solution of the following integral equation

$$\begin{aligned}
(6.29) \quad z^{(k)}(t, x; h^{(k)}) &= z^{*(k)}(\tau_-^{(k)}(t, x, h^{(k)}), X^{(k)}(\tau_-^{(k)}(t, x, h^{(k)}); t, x, h^{(k)})) + \\
& + \int_{\tau_-^{(k)}(t, x, h^{(k)})}^t \left[-c^{(k)}(s, X^{(k)}(s; t, x, h^{(k)}), h^{(k)}(s, X^{(k)}(s; t, x, h^{(k)}))) \times \right. \\
& \times z^{(k)}(s, X^{(k)}(s; t, x, h^{(k)}); h^{(k)}) + a^{(k)}(s, X^{(k)}(s; t, x, h^{(k)}), h^{(k)}(s, X^{(k)}(s; t, x, h^{(k)}))) \left. \right] ds,
\end{aligned}$$

where $X^{(k)}(\cdot; t, x, h^{(k)}) : [\tau_-^{(k)}(t, x, h^{(k)}), \tau_+^{(k)}(t, x, h^{(k)})] \rightarrow G$ is the flow associated to the vector field $b^{(k)}$ with $k = 1, 2$, then for every $M > 0$ there exists $\delta > 0$ depending on $h^{(1)}, b^{(1)}, M$, such that, under the conditions $\|h^{(2)} - h^{(1)}\|_\infty < \delta$, $\|b^{(2)} - b^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} < \delta$ and $\|b^{(k)}\|_{L^\infty(0, t_1; h^{(k)})} \leq M$ with $k = 1, 2$, we have

$$\begin{aligned}
(6.30) \quad & \|z^{(2)}(\cdot; h^{(2)}) - z^{(1)}(\cdot; h^{(1)})\|_{L^\infty(S_{t_1})} \leq \\
& \leq C_9 \left[\|z^{*(2)} - z^{*(1)}\|_{L^\infty(\Gamma_-)} + \|b^{(2)} - b^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} + \|c^{(2)} - c^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} + \right. \\
& \quad \left. + \|a^{(2)} - a^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} + \|h^{(2)} - h^{(1)}\|_\infty \right],
\end{aligned}$$

where

$$(6.31) \quad C_9 = C \left(1 + \bigvee_{k=1,2} \|b^{(k)}\|_{L^\infty(0, t_1; h^{(k)})} \right) \left(1 + \|D_{(x,y)} b^{(1)}\|_{L^1(0, t_1; h^{(1)}, h^{(2)})} \right) \times$$

$$\begin{aligned}
& \times \bigvee_{k=1,2} \left[1 + L^{*(k)2} + \|z^{*(k)}\|_{L^\infty(\Gamma_-)}^2 + \|c^{(k)}\|_{L^\infty(0,t_1;h^{(k)})}^2 + \|a^{(k)}\|_{L^\infty(0,t_1;h^{(k)})}^2 + \right. \\
& + \Lambda (D_x h^{(1)}, D_x h^{(2)})^2 (\|\nabla_{(x,y)} c^{(k)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})}^2 + \|\nabla_{(x,y)} a^{(k)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})}^2) \Big] \times \\
& \times \exp \left[C \left(\bigvee_{k=1,2} [\|b^{(k)}\|_{L^\infty(0,t_1;h^{(k)})} + \|c^{(k)}\|_{L^\infty(0,t_1;h^{(k)})}] + \right. \right. \\
& \left. \left. + \Lambda (D_x h^{(1)}, D_x h^{(2)}) \|D_{(x,y)} b^{(1)}\|_{L^1(0,t_1;h^{(1)},h^{(2)})} \right) \right]
\end{aligned}$$

where the constant C depends only on $B_d^{(k)}$, $\|b_d^{(k)}\|_{L_{x_d}^1(0,1;L_{(t,x')}^\infty(S'_{t_1}))}$, $\|D_{(t,x')} b_d^{(1)}\|_{L_{x_d}^1(0,1;L_{(t,x')}^\infty(S'_{t_1}))}$ and $L^{*(k)}$ is the Lipschitz constant for $z^{*(k)}$ (with $k = 1, 2$).

Proof.

Taking into account Lemma 5.3, the proof is analogous to the one used in Lemma 6.2.

□

7 - The proof of the main theorem.

Before to prove the main theorem of this paper, we give a preliminary lemma about the semilinear part of our quasilinear hyperbolic system.

Lemma 7.1. Assume (3.11), (3.12) and (3.14) for f_i , v_i and y_{0i} ($i = 1, \dots, p$) respectively. If C_W is a positive constant, then there exists $0 < t_2(C_W) \leq t_1$ such that the following system of integral equations

(7.1)

$$y_i(t, x) = y_{0i}(X_{iY}(0; t, x)) + \int_0^t f_i(s, X_{iY}(s; t, x), y(s, X_{iY}(s; t, x), \bar{w}(s, X_{iY}(s; t, x)))) ds,$$

$$(7.2) \quad X_{iY}(s; t, x) = x - \int_s^t v_i(r, X_{iY}(r; t, x)) dr, \quad s \in [0, t],$$

where $i = 1, \dots, p$ and $\|\bar{w}\|_{L^\infty(0,t_1;W^{1,\infty}(G))} \leq C_W$, admits one and only one solution $y \in L^\infty(0, t_2; W^{1,\infty}(G))^p$ which satisfies the following estimates

$$(7.3) \quad \|y\|_{L^\infty(S_{t_2})} \leq \|y_0\|_{L^\infty(G)} + 1, \quad \|D_x y\|_{L^\infty(S_{t_2})} \leq 2pd \|D_x y_0\|_{L^\infty(G)} + 1.$$

Furthermore, if $\bar{w}^{(k)}$ satisfies

$$(7.4) \quad \|\bar{w}^{(k)}\|_{L^\infty(0,t_1;W^{1,\infty}(G))} \leq C_W$$

and $y^{(k)} \in L^\infty(0, t_2; W^{1,\infty}(G))^p$ is the solution of (7.1)-(7.2) with $\bar{w} = \bar{w}^{(k)}$ where $k = 1, 2$, then we have

$$(7.5) \quad \|y^{(2)} - y^{(1)}\|_{L^\infty(S_{t_2})} \leq \exp(\|D_y f\|_{L^1(0,t_2;(y^{(1)}, \bar{w}^{(1)}), (y^{(2)}, \bar{w}^{(2)}))}) \times$$

$$\times \|D_w f\|_{L^1(0,t_2;(y^{(1)},\bar{w}^{(1)}),(y^{(2)},\bar{w}^{(2)}))} \|\bar{w}^{(2)} - \bar{w}^{(1)}\|_{L^\infty(S_{t_2})}.$$

Proof.

First we consider the following linearized version of (7.1)
(7.6)

$$\tilde{y}_i(t, x) = y_{0i}(X_{iY}(0; t, x)) + \int_0^t f_i(s, X_{iY}(s; t, x), \bar{y}(s, X_{iY}(s; t, x), \bar{w}(s, X_{iY}(s; t, x)))ds,$$

where $i = 1, \dots, p$, $\bar{y} \in L^\infty(0, t_1; W^{1,\infty}(G))^p$ and it satisfies the following estimates

$$(7.7) \quad \|\bar{y}\|_{L^\infty(S_{t_1})} \leq \|y_0\|_{L^\infty(G)} + 1, \quad \|D_x \bar{y}\|_{L^\infty(S_{t_1})} \leq 2pd \|D_x y_0\|_{L^\infty(G)} + 1.$$

It is obvious that the problem (7.6)-(7.2) admits one and only one solution $\tilde{y} \in L^\infty(0, t_1; W^{1,\infty}(G))^p$ explicitly defined by (7.6). Therefore, we can consider the following operator $L_t : L^\infty(0, t; W^{1,\infty}(G))^p \rightarrow L^\infty(0, t; W^{1,\infty}(G))^p$ such that $L_t(\bar{y}) = \tilde{y}$ with $0 < t \leq t_1$. To show that L_t is a contraction for t sufficiently small, we need some estimates about \tilde{y} . More precisely, from (7.6)-(7.2), we obtain

$$(7.8) \quad \|\tilde{y}\|_{L^\infty(S_t)} \leq \|y_0\|_{L^\infty(G)} + \|f\|_{L^1(0,t;\bar{y},\bar{w})}, \quad \|D_x X_Y\|_{L^\infty(S_t)} \leq pd \exp(\|D_x v\|_{L^1(0,t;L^\infty(G))}),$$

where $0 < t \leq t_1$ and $X_Y = (X_{1Y}, \dots, X_{pY})$. Now, applying the differential operator D_x to (7.6) and taking into account (7.8), we obtain the following estimates

$$(7.9) \quad \|D_x \tilde{y}\|_{L^\infty(S_t)} \leq (pd) \exp(\|D_x v\|_{L^1(0,t;L^\infty(G))}) (\|D_x y_0\|_{L^\infty(G)} + \|D_x f\|_{L^1(0,t;\bar{y},\bar{w})} + \\ + \|D_y f\|_{L^1(0,t;\bar{y},\bar{w})} \|D_x \bar{y}\|_{L^\infty(S_t)} + \|D_w f\|_{L^1(0,t;\bar{y},\bar{w})} \|D_x \bar{w}\|_{L^\infty(S_t)}), \quad 0 < t \leq t_1.$$

Hence, from (7.7), (7.8) and (7.9), we deduce that there exists $0 < t'_1 \leq t_1$ such that

$$(7.10) \quad \|\tilde{y}\|_{L^\infty(S_t)} \leq \|y_0\|_{L^\infty(G)} + 1, \quad \|D_x \tilde{y}\|_{L^\infty(S_t)} \leq 2pd \|D_x y_0\|_{L^\infty(G)} + 1, \quad 0 < t \leq t'_1;$$

therefore $L_t(C_t) \subseteq C_t$ if we define C_t as follows

$$(7.11) \quad C_t = \{z \in L^\infty(0, t; W^{1,\infty}(G))^p |$$

$$\|z\|_{L^\infty(S_t)} \leq \|y_0\|_{L^\infty(G)} + 1, \quad \|D_x z\|_{L^\infty(S_t)} \leq 2pd \|D_x y_0\|_{L^\infty(G)} + 1\}, \quad 0 < t \leq t'_1.$$

Hence C_t is a closed of $L^\infty(0, t; W^{1,\infty}(G))^p$. Thus, we have to show that $L_t : C_t \rightarrow C_t$ is a contraction; for this purpose, assuming $\bar{y} = \bar{y}^{(k)} \in C_t$ and denoting by $\tilde{y}^{(k)} \in C_t$ the solution of the problem (7.6)-(7.2) with $k = 1, 2$, it is not so hard to obtain the following perturbation estimate

$$(7.12) \quad \|\tilde{y}^{(2)} - \tilde{y}^{(1)}\|_{L^\infty(S_t)} \leq \|D_y f\|_{L^1(0,t;(\bar{y}^{(1)},\bar{w}),(\bar{y}^{(2)},\bar{w}))} \|\bar{y}^{(2)} - \bar{y}^{(1)}\|_{L^\infty(S_t)}.$$

Therefore there exists $0 < t_2 \leq t'_1$ such that L_t is a contraction; consequently, thanks to Banach-Caccioppoli's fixed point theorem, L_t admits one and only one fixed point denoted by y . Of course y is the solution of the problem (7.1)-(7.2).

To prove (7.5), it is enough to consider the integral expressions for $y^{(1)}$ and $y^{(2)}$, to make the difference between them and finally to apply Gronwall's lemma.

□

Remark 7.1.0. Of course, in Lemma 7.1, the solution y lies in $W^{1,\infty}(S_{t_2})^p$. However, in the proof of the main theorem, first, we prove that the solution (y, w) of our IBVP belongs to $L^\infty(0, t^*; W^{1,\infty}(G))^{p+q}$; afterwards, it is straightforward to deduce $(y, w) \in W^{1,\infty}(S_{t^*})^{p+q}$.

Now we are ready to prove Theorem 3.1.

Proof of the main theorem.

Let us define the following vectors

$$(7.13) \quad \alpha = \frac{1}{p}(\|y_0\|_{L^\infty(G)} + 1) \sum_{i=1}^p e_i^{(p)}, \quad \beta = \frac{1}{q}(\|w^*\|_{L^\infty(\Gamma_-)} + 1) \sum_{j=1}^q e_j^{(q)}$$

where $\{e_1^{(p)}, \dots, e_p^{(p)}\}$, $\{e_1^{(q)}, \dots, e_q^{(q)}\}$ are the canonical bases of \mathbb{R}^p and \mathbb{R}^q respectively; of course, we have

$$(7.14) \quad \|\alpha\|_{L^\infty(S_{t_1})} = \|y_0\|_{L^\infty(G)} + 1, \quad \|\beta\|_{L^\infty(S_{t_1})} = \|w^*\|_{L^\infty(\Gamma_-)} + 1.$$

Moreover, in relation to the vector fields $u_j \in L_{x_d}^-(S_{t_1})$, we introduce the positive constants B_{jd} as follows

$$(7.15) \quad u_{jd}(t, x) \leq -B_{jd} \quad \text{a.e. } t \in (0, t_1), \quad \forall x \in G.$$

Afterwards, we define the following closed subset of $L^\infty(0, t; W^{1,\infty}(G))^q$

$$(7.16) \quad K_t = \{z \in L^\infty(0, t; W^{1,\infty}(G))^q\}$$

$$\begin{aligned} & \|z\|_{L^\infty(S_t)} \leq \|w^*\|_{L^\infty(\Gamma_-)} + 1, \quad \|D_x z\|_{L^\infty(S_t)} \leq C \left(2 + \|u\|_{L^\infty(0,t;\alpha)}^2\right) \times \\ & \times \left(2 + L^{*2} + \|w^*\|_{L^\infty(\Gamma_-)}^2 + \|g\|_{L^\infty(0,t;\alpha,\beta)}^2\right) \exp C \left(1 + \|u\|_{L^\infty(0,t;\alpha)}\right), \quad 0 < t \leq t_1; \end{aligned}$$

where C is the same constant that appears in the definition of C_6 (see (6.17)) but in this case it depends on $\sum_{j=1}^q B_{jd}$, $\sum_{j=1}^q \|D_{(t,x')} u_{jd}\|_{L_{x_d}^1(0,1;L_{(t,x')}^\infty(S'_{t_1}))}$; moreover, L^* is the lipschitz constant for w^* . Thanks to Lemma 7.1, there exists $0 < t_2 \leq t_1$ such that Cauchy's problem (7.1)-(7.2) admits one and only one solution $y \in L^\infty(0, t_2; W^{1,\infty}(G))^p$ for every $\bar{w} \in K_{t_1}$. Now, we consider the operator $H_t : K_t \rightarrow L^\infty(0, t; W^{1,\infty}(G))^q$ such that $H_t(\bar{w}) = \tilde{w}$ where $0 < t \leq t_2$, \tilde{w} is defined as follows

$$(7.17) \quad \tilde{w}_j(t, x) = w_j^*(\tau_{j-}(t, x, y), X_{jW}(\tau_{j-}(t, x, y); t, x, y)) +$$

$$+ \int_{\tau_{j-}(t,x,y)}^t g_j(s, X_{jW}(s; t, x, y), y(s, X_{jW}(s; t, x, y)), \bar{w}(s, X_{jW}(s; t, x, y))) ds,$$

$$(7.18)$$

$$X_{jW}(s; t, x, y) = x - \int_s^t u_j(r, X_{jW}(r; t, x, y), y(r, X_{jW}(r; t, x, y))) dr, \quad s \in [\tau_{j-}(t, x, y), t],$$

$\bar{w} \in K_t$ and $j = 1, \dots, q$.

With reference to Lemma 6.2, fixed $\bar{w} \in K_t$ and j , we put

$$(7.19) \quad z = \tilde{w}_j, \quad z^* = w_j^*, \quad c = 0, \quad h = \bar{h} = y, \quad a(\cdot, h(\cdot)) = g_j(\cdot, y(\cdot), \bar{w}(\cdot)), \quad b = u_j;$$

therefore the estimates (6.5) and (6.16) added from $j = 1$ to $j = q$, give us the following bounds for \tilde{w}

$$(7.20) \quad \|\tilde{w}\|_{L^\infty(S_t)} \leq \|w^*\|_{L^\infty(\Gamma_-)} + \|g\|_{L^1(0,t;y,\bar{w})},$$

$$(7.21) \quad \|D_x \tilde{w}\|_{L^\infty(S_t)} \leq C \left(1 + \|u\|_{L^\infty(0,t;y)}^2 + \|D_{(x,y)} u\|_{L^1(0,t;y)}^2 \right) \left[1 + L^{*2} + \|w^*\|_{L^\infty(\Gamma_-)}^2 + \right. \\ \left. + \|g\|_{L^\infty(0,t;y,\bar{w})}^2 + \Lambda (D_x y, D_x y)^2 \left(\|D_{(x,y)} g\|_{L^1(0,t;y,\bar{w})} + \|D_w g\|_{L^1(0,t;y,\bar{w})} \|D_x \bar{w}\|_{L^\infty(S_t)} \right)^2 \right] \times \\ \times \exp \left[C \left(\|u\|_{L^\infty(0,t;y)} + \Lambda (D_x y, D_x y) \|D_{(x,y)} u\|_{L^1(0,t;y)} \right) \right].$$

From these estimates we immediately deduce there exists $0 < t'_2 \leq t_2$ such that if $0 < t \leq t'_2$ then we have $H_t(K_t) \subseteq K_t$. Hence, to prove the existence and uniqueness of the solution for the system of integral equations (3.7)-(3.10), we must prove that H_t is a contraction (for small values of t). For this purpose, let $\bar{w}^{(k)}$ be given in K_t with $0 < t \leq t'_2$ and $k = 1, 2$; moreover, we indicate the solution of (7.1)-(7.2), with $\bar{w} = \bar{w}^{(k)}$, by $y^{(k)} \in L^\infty(0, t; W^{1,\infty}(G))^p$. Afterwards, we define $\tilde{w}^{(k)} = H_t(\bar{w}^{(k)})$. Now, with reference to Lemma 6.4, we put

$$(7.22) \quad z^{(k)} = \tilde{w}_j^{(k)}, \quad z^{*(k)} = w_j^*, \quad c^{(k)} = 0, \quad h^{(k)} = y^{(k)}, \\ a^{(k)}(\cdot, h^{(k)}(\cdot)) = g_j(\cdot, y^{(k)}(\cdot), \bar{w}^{(k)}(\cdot)), \quad b^{(k)} = u_j.$$

Furthermore, thanks to the estimate (7.5), there exists $0 < t_3 \leq t'_2$ such that

$$(7.23) \quad \|y^{(2)} - y^{(1)}\|_{L^\infty(S_{t_3})} < \delta$$

where δ is defined as in Lemma 6.4; therefore the estimate (6.30) added from $j = 1$ to $j = q$, give us the following perturbation estimate

$$(7.24) \quad \|\tilde{w}^{(2)} - \tilde{w}^{(1)}\|_{L^\infty(S_t)} \leq C \left\{ \|g(\cdot, \bar{w}^{(2)}) - g(\cdot, \bar{w}^{(1)})\|_{L^1(0,t;y^{(1)},y^{(2)})} + \|y^{(2)} - y^{(1)}\|_{L^\infty(S_t)} \right\}$$

where $0 < t \leq t_3$ and the constant C does not depend from $\bar{w}^{(k)}$ with $k = 1, 2$. Now, using (7.5) and simple manipulations we deduce

$$(7.25) \quad \|\tilde{w}^{(2)} - \tilde{w}^{(1)}\|_{L^\infty(S_t)} \leq C \left[\|D_w g\|_{L^1(0,t;(y^{(1)},\bar{w}^{(1)}),(y^{(2)},\bar{w}^{(2)}))} + \right. \\ \left. + \exp(\|D_y f\|_{L^1(0,t;(y^{(1)},\bar{w}^{(1)}),(y^{(2)},\bar{w}^{(2)}))}) \|D_w f\|_{L^1(0,t;(y^{(1)},\bar{w}^{(1)}),(y^{(2)},\bar{w}^{(2)}))} \right] \times \\ \times \|\bar{w}^{(2)} - \bar{w}^{(1)}\|_{L^\infty(S_t)}, \quad 0 < t \leq t_3.$$

Hence there exists $0 < t^* \leq t_3$ such that Λ_{t^*} is a contraction, therefore, thanks to Banach-Caccioppoli's fixed point theorem, this operator admits one and only one fixed point w .

Therefore we have showed that (y, w) , where y is the solution of (7.1)-(7.2) with $\bar{w} = w$, is the generalized solution of (3.1)-(3.4).

Afterwards we study the continuous dependence of generalized solutions on data for the problem (3.1)-(3.4). For this purpose, we assume (3.1)-(3.4) :

$$(7.26) \quad f_i^{(k)}, g_j^{(k)} \in L_t^\infty(0, t_1; L_{(x, y_{loc}, w_{loc})}^\infty(G \times \mathbb{R}^p \times \mathbb{R}^q)) \cap L_t^1(0, t_1; W_{(x, y_{loc}, w_{loc})}^{1, \infty}(G \times \mathbb{R}^p \times \mathbb{R}^q)),$$

$$(7.27) \quad v_i^{(k)} \in L^1(0, t_1; W^{1, \infty}(G))^d, \quad u_j^{(k)} \in L_{x_d}^-(S_{t_1} \times \mathbb{R}^p),$$

$$(7.28) \quad v_i^{(k)}(t, x', 0) \cdot e_d = v_i^{(k)}(t, x', 1) \cdot e_d = 0 \quad \forall x' \in \mathbb{R}^{d-1}, \text{ a.e. } t \in (0, t_1),$$

$$(7.29) \quad y_{0i}^{(k)} \in W^{1, \infty}(G), \quad w_j^{*(k)} \in Lip_{loc}^{unif}(\Gamma_-), \quad i = 1, \dots, p, \quad j = 1, \dots, q, \quad k = 1, 2.$$

With reference to (3.7)-(3.10), we assume $v_i = v_i^{(k)}$, $f_i = f_i^{(k)}$, $y_{0i} = y_{0i}^{(k)}$, $u_j = u_j^{(k)}$, $g_j = g_j^{(k)}$, $w_j^* = w_j^{*(k)}$ and we denote by $(y_i^{(k)}, w_j^{(k)})$ the solution of (3.7)-(3.10) in $(0, \tilde{t})$ with $i = 1, \dots, p$, $j = 1, \dots, q$ and $k = 1, 2$. Subtracting (3.7), (3.9) with $k = 2$ from the same equations with $k = 1$, we obtain, after some calculations, the following estimate

$$(7.30) \quad \begin{aligned} & \|y^{(2)}(t, \cdot) - y^{(1)}(t, \cdot)\|_{L^\infty(G)} \leq \|y_0^{(2)} - y_0^{(1)}\|_{L^\infty(G)} + \|f^{(2)} - f^{(1)}\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} + \\ & + (\|D_x y_0^{(1)}\|_{L^\infty(G)} + \|D_x f^{(1)}\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} + \|D_y f^{(1)}\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} \times \\ & \quad \times \|D_x y^{(1)}\|_{L^\infty(S_t)} + \|D_w f^{(1)}\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} \|D_x w^{(1)}\|_{L^\infty(S_t)}) \times \\ & \quad \times \|v^{(2)} - v^{(1)}\|_{L^1(0, t; L^\infty(G))} \exp \|D_x v^{(1)}\|_{L^1(0, t; L^\infty(G))} + \\ & + \int_0^t \|D_y f^{(1)}(s)\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} \|y^{(2)}(s, \cdot) - y^{(1)}(s, \cdot)\|_{L^\infty(G)} ds + \\ & + \int_0^t \|D_w f^{(1)}(s)\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} \|w^{(2)}(s, \cdot) - w^{(1)}(s, \cdot)\|_{L^\infty(G)} ds. \end{aligned}$$

Now, with reference to Lemma 6.4, we assume

$$(7.31) \quad \begin{aligned} z^{(k)} &= w_j^{(k)}, \quad z^{*(k)} = w_j^{*(k)}, \quad c^{(k)} = 0, \quad h^{(k)} = y^{(k)}, \\ a^{(k)}(\cdot, h^{(k)}(\cdot)) &= g_j^{(k)}(\cdot, y^{(k)}(\cdot), w^{(k)}(\cdot)), \quad b^{(k)} = u_j^{(k)}; \end{aligned}$$

therefore, applying (6.30) and adding it from $j = 1$ to q , we obtain

$$(7.32) \quad \begin{aligned} & \|w^{(2)}(t, \cdot) - w^{(1)}(t, \cdot)\|_{L^\infty(G)} \leq C \left(\|w^{*(2)} - w^{*(1)}\|_{L^\infty(\Gamma_-)} + \|u^{(2)} - u^{(1)}\|_{L^1(0, t; y^{(1)}, y^{(2)})} + \right. \\ & \quad \left. + \|g^{(2)} - g^{(1)}\|_{L^1(0, t; (y^{(1)}, w^{(1)}), (y^{(2)}, w^{(2)}))} + \|y^{(2)} - y^{(1)}\|_{L^\infty(S_t)} \right). \end{aligned}$$

Hence, combining (7.30) with (7.32), we deduce the result of continuous dependence.

Finally, remembering Corollary 6.3 and Remark 7.1.0, we deduce that $(y, w) \in W^{1, \infty}(S_{t^*}^{p+q})$.

□

8 - The well-posedness of an IBVP for the hyperbolic part of an atmospheric model.

In the paper [28] we have introduced and studied a model of the phase transitions for H_2O in the three states in the atmosphere. Let us say something about this model. We can consider a spatial domain Ω in the atmosphere. Inside Ω there are many particles interacting among them characterized by densities and velocities. More exactly, we consider the density ρ of dry air, the density π of water vapour, the density σ of H_2O in liquid state, the density ν of H_2O in solid state, the speed v and the temperature T of the atmospheric gas, the speed u of water droplets and the speed w of ice crystals. In the model proposed in [28], these unknown quantities are linked among them by six partial differential equations, of which four of them are hyperbolic type whereas the others are parabolic type, whereas the speed u of droplets and the speed w of ice crystals are given by simple formulas involving v .

In this section, using the main theorem of this paper, we study IBVP for the hyperbolic part of the model in [28] on the strip $\Omega_M = \{(m, x_1, x_2, x_3) \mid m, x_1, x_2 \in \mathbb{R}, 0 < x_3 < 1\}$ with given velocities v, u, w , and temperature T . Moreover, we assume v to be tangent to the planes $x_3 = 0$ and $x_3 = 1$, whereas u and w have negative vertical components; therefore, rain and ice fall from the strip. Hence we consider the following system of partial differential equations in the unknown functions ρ, π, σ, ν

$$(8.1) \quad \partial_t \rho(t, x) + v(t, x) \cdot \nabla_x \rho(t, x) = R^*(\rho)(t, x),$$

$$(8.2) \quad \partial_t \pi(t, x) + v(t, x) \cdot \nabla_x \pi(t, x) = P^*(\pi, \sigma, \nu)(t, x),$$

$$(8.3) \quad \begin{aligned} \partial_t \sigma(t, m, x) + (s_l(m) [(\pi - \pi_l(T))(t, x)], u(t, m, x)) \cdot \nabla_{(m, x)} \sigma(t, m, x) = \\ = S^*(\pi, \sigma, \nu)(t, m, x), \end{aligned}$$

$$(8.4) \quad \begin{aligned} \partial_t \nu(t, m, x) + (s_s(m) [(\pi - \pi_s(T))(t, x)], w(t, m, x)) \cdot \nabla_{(m, x)} \nu(t, m, x) = \\ = N^*(\pi, \sigma, \nu)(t, m, x), \end{aligned}$$

where $(t, m, x) \in S_{t_1} = (0, t_1) \times \mathbb{R} \times \mathbb{R}^2 \times (0, 1) = (0, t_1) \times \Omega_M = (0, t_1) \times \mathbb{R} \times \Omega = S'_{t_1} \times (0, 1)$ and $Q_{t_1} = (0, t_1) \times \Omega$; moreover we assume the following conditions

$$(8.5) \quad \rho(0, x) = \rho_0(x) \quad x \in \Omega,$$

$$(8.6) \quad \pi(0, x) = \pi_0(x) \quad x \in \Omega,$$

$$(8.7) \quad \sigma(t, m, x) = \sigma^*(t, m, x) \quad (t, m, x) \in \Gamma_-,$$

$$(8.8) \quad \nu(t, m, x) = \nu^*(t, m, x) \quad (t, m, x) \in \Gamma_-,$$

where Γ_- is the surface carrying data defined as $(\{0\} \times \Omega_M) \cup ([0, t_1] \times \mathbb{R}^3 \times \{1\})$ and $\rho_0, \pi_0, \sigma^*, \nu^*$ are given functions.

Let us recall the quantities which figure in IBVP (8.1)-(8.8). The variable t is the time, m is the mass of a water droplet or an ice crystal, whereas x is the position vector; in particular, x_3 is the height of a generic point identified by x . Moreover, knowing that there are no droplets or ice crystals with smaller mass than $m_a > 0$ (see [28]), it will not be restrictive to extend the functions in the partial differential equations of our IBVP to zero for $m \leq 0$; this makes it possible to assume $m \in \mathbb{R}$.

The quantities ρ_0, π_0 are initial densities (for $t = 0$), whereas σ^*, ν^* are the prescribed densities on Γ_- .

Now, we define the functions appearing in the second members of (8.1)-(8.4)

$$(8.9) \quad R^*(\rho)(t, x) = -[(\nabla_x \cdot v)\rho](t, x);$$

$$(8.10) \quad P^*(\pi, \sigma, \nu)(t, x) = -[(\nabla_x \cdot v)\pi + P(\pi, \sigma, \nu)](t, x),$$

where $P(\pi, \sigma, \nu)$ represents the total amount of water vapour that is transformed into liquid or solid state and we assume that :

$$(8.11) \quad P(\pi, \sigma, \nu)(t, x) = -[(\pi - \pi_l(T))F_l(\sigma) - (\pi - \pi_s(T))F_s(\nu)](t, x),$$

$$(8.12) \quad F_l(\sigma)(t, x) = \int_0^\infty \bar{s}_l(m) \sigma(t, m, x) dm,$$

$$(8.13) \quad F_s(\nu)(t, x) = \int_0^\infty \bar{s}_s(m) \nu(t, m, x) dm;$$

$$(8.14) \quad \begin{aligned} S^*(\pi, \sigma, \nu)(t, m, x) = & -[(\nabla_{(x,m)} \cdot u)\sigma](t, m, x) + \\ & + [S_g(\pi, \sigma) + S_s(\sigma, \nu) + S_a(\pi, \sigma, \nu) + S_q(\sigma, \nu)](t, m, x), \end{aligned}$$

where

$$(8.15) \quad S_g(\pi, \sigma)(t, m, x) = \bar{s}_l(m) [\pi(t, x) - \pi_l(T(t, x))]\sigma(t, m, x)$$

is the amount of H_2O converted from gas to liquid that condenses on droplets with mass m ,

$$(8.16) \quad S_s(\sigma, \nu)(t, m, x) = [-K_{ls}(m, T)\sigma(m) + K_{sl}(m, T)\nu(m)](t, x)$$

is the amount of droplets with mass m that appears or disappears due to the solidification or fusion,

$$(8.17) \quad S_a(\pi, \sigma, \nu)(t, m, x) =$$

$$\begin{aligned}
&= g_a(m) \left[N^*(t, x) - \int_0^{+\infty} n_l(m) \sigma(t, m, x) dm - \int_0^{+\infty} n_s(m) \nu(t, m, x) dm \right]^+ [\pi(t, x) - \pi_l(T(t, x))]^+ - \\
&\quad - g_l(m) [\pi(t, x) - \pi_l(T(t, x))]^- \sigma(t, m, x)
\end{aligned}$$

is the total amount of droplets with mass m which arises or evaporates on the aerosol particles,

$$(8.18) \quad S_q(\sigma, \nu)(t, m, x) = [Q_l(\sigma, \sigma) + J_l(\sigma)\sigma + J_{ls}(\nu)\sigma](t, m, x)$$

is relative to interactions among droplets and ice crystals and we give the following additional definitions

$$(8.19) \quad J_l(\sigma)(t, m, x) = -m \int_0^\infty \beta_l(m, m') \sigma(t, m', x) dm',$$

$$(8.20) \quad Q_l(\sigma, \sigma')(t, m, x) = \frac{m}{2} \int_0^m \beta_l(m', m - m') \sigma(t, m', x) \sigma'(t, m - m', x) dm',$$

$$(8.21) \quad J_{ls}(\omega)(t, m, x) = -m \int_0^\infty Z_{ls}(m', m) \omega(t, m', x) dm', \quad \omega = \nu, \sigma,$$

where $\beta_l(m, m')$ is related to the probability that droplets of mass m and m' collide and merge, whereas $Z_{ls}(m, m')$ regards the probability that a droplet of mass m' joins an ice particle of mass m (with instantaneous phase transition from the liquid to the solid state);

$$\begin{aligned}
(8.22) \quad N^*(\pi, \sigma, \nu)(t, m, x) &= -[(\nabla_{(x, m)} \cdot w) \nu](t, m, x) + \\
&+ [N_g(\pi, \nu) + N_a(\pi, \nu) + N_s(\sigma, \nu) + N_q(\sigma, \nu)](t, m, x),
\end{aligned}$$

where

$$(8.23) \quad N_g(\pi, \nu)(t, m, x) = \bar{s}_s(m) [\pi(t, x) - \pi_s(T(t, x))] \nu(t, m, x),$$

$$(8.24) \quad N_s(\sigma, \nu)(t, m, x) = [K_{ls}(m, T) \sigma(m) - K_{sl}(m, T) \nu(m)](t, x),$$

$$(8.25) \quad N_a(\pi, \nu)(t, m, x) = -g_s(m) [\pi(t, x) - \pi_s(T(t, x))]^- \nu(t, m, x),$$

$$(8.26) \quad N_q(\sigma, \nu)(t, m, x) = [Q_s(\nu, \nu) + J_s(\nu)\nu + Q_{ls}(\nu, \sigma) + J_{ls}(\sigma)\nu](t, m, x),$$

$$(8.27) \quad J_s(\nu)(t, m, x) = -m \int_0^\infty \beta_s(m, m') \nu(t, m', x) dm',$$

$$(8.28) \quad Q_s(\nu, \nu')(t, m, x) = \frac{m}{2} \int_0^m \beta_l(m', m - m') \nu(t, m', x) \nu'(t, m - m', x) dm',$$

$$(8.29) \quad Q_{ls}(\nu, \sigma)(t, m, x) = \frac{m}{2} \int_0^m Z_{ls}(m', m - m') \sigma(t, m', x) \nu(t, m - m', x) dm'.$$

These last quantities are defined in a similar way as already we have seen above about droplets. Moreover, it is possible to find the definitions about the physical quantities such as s_j , \bar{s}_j , π_j , N^* , n_j , g_a , g_j , K_{ls} and K_{sl} ($j = l, s$) in [28], [2], etc.

To the initial and boundary value problem (8.1) - (8.8) we can associate, by the method of characteristics, the following system of integral equations

$$(8.30) \quad \rho(t, x) = \rho_0(X_\Pi(0; t, x)) + \int_0^t R^*(\rho)(r, X_\Pi(r; t, x)) dr,$$

$$(8.31) \quad \pi(t, x) = \pi_0(X_\Pi(0; t, x)) + \int_0^t P^*(\pi, \sigma, \nu)(r, X_\Pi(r; t, x)) dr,$$

$$(8.32) \quad \begin{aligned} \sigma(t, m, x) = & \sigma^*(\tau_{\Sigma-}(t, m, x; \pi), X_\Sigma(\tau_{\Sigma-}(t, m, x; \pi); t, m, x; \pi)) + \\ & + \int_{\tau_{\Sigma-}(t, m, x; \pi)}^t S^*(\pi, \sigma, \nu)(r, X_\Sigma(r; t, m, x; \pi)) dr, \end{aligned}$$

$$(8.33) \quad \begin{aligned} \nu(t, m, x) = & \nu^*(\tau_{N-}(t, m, x; \pi), X_N(\tau_{N-}(t, m, x; \pi); t, m, x; \pi)) + \\ & + \int_{\tau_{N-}(t, m, x; \pi)}^t N^*(\pi, \sigma, \nu)(r, X_N(r; t, m, x; \pi)) dr, \end{aligned}$$

where $(t, m, x) \in S_{t_1}$, the fluxes X_Π , X_Σ , X_N are defined as follows

$$(8.34) \quad X_\Pi(r; t, x) = x + \int_t^r v(q, X_\Pi(q; t, x)) dq, \quad r \in [0, t_1],$$

$$(8.35) \quad \begin{aligned} X_\Sigma(r; t, m, x; \pi) = & (m, x) + \int_t^r (s_l(\pi - \pi_l(T)), u)(q, X_\Sigma(q; t, m, x; \pi)) dq, \\ & r \in [\tau_{\Sigma-}(t, m, x; \pi), t], \end{aligned}$$

$$(8.36) \quad \begin{aligned} X_N(r; t, m, x; \pi) = & (m, x) + \int_t^r (s_s(\pi - \pi_s(T)), w)(q, X_N(q; t, m, x; \pi)) dq, \\ & r \in [\tau_{N-}(t, m, x; \pi), t], \end{aligned}$$

and $\tau_{\Sigma-}(t, m, x; \pi)$, $\tau_{N-}(t, m, x; \pi)$ are the minimal time of existence for the solutions of (8.35), (8.36) respectively.

Moreover any solution of the integral equations system (8.30)-(8.36) will be considered as a generalized solution for the IBVP (8.1)-(8.8).

We make now assumptions on the functions that appear in (8.1)-(8.4). More exactly, we assume the following conditions on T , v , u and w :

$$(8.37) \quad T \in L^\infty(0, t_1; L^\infty(\Omega)) \cap L^1(0, t_1; W^{1,\infty}(\Omega)) ,$$

$$(8.38) \quad v \in L^\infty(0, t_1; L^\infty(\Omega))^3 \cap L^1(0, t_1; W^{1,\infty}(\Omega))^3 ,$$

$$(8.39) \quad v(t, x_1, x_2, 0) \cdot e_3 = v(t, x_1, x_2, 1) \cdot e_3 = 0, \quad \forall (x_1, x_2) \in \mathbb{R}^2 \quad \text{for almost every } t \in (0, t_1)$$

($\{e_k | k = 1, 2, 3\}$ is the canonical frame on \mathbb{R}^3),

$$(8.40) \quad u, w \in L^\infty(0, t_1; L^\infty(\Omega_M))^3 \cap L^1(0, t_1; W^{1,\infty}(\Omega_M))^3 ,$$

$$(8.41) \quad u_3, w_3 \in L^1_{x_3}(0, 1; W^{1,\infty}_{(t,m,x_1,x_2)}(S'_{t_1})),$$

$$(8.42) \quad u_3(t, m, x), w_3(t, m, x) \leq -B \quad \forall (m, x) \in \Omega_M \text{ a.e. } t \in (0, t_1),$$

where B is a positive constant (we say that $u, w \in \tilde{L}^-_{x_3}(S_{t_1})$ for reasons of convenience),

$$(8.43) \quad \nabla_x \cdot v \in L^\infty(0, t_1; L^\infty(\Omega)) \cap L^1(0, t_1; W^{1,\infty}(\Omega)) ,$$

$$(8.44) \quad \nabla_{(m,x)} \cdot u, \nabla_{(m,x)} \cdot w \in L^\infty(0, t_1; L^\infty(\Omega_M)) \cap L^1(0, t_1; W^{1,\infty}(\Omega_M)) .$$

As for initial and boundary conditions, we take

$$(8.45) \quad \rho_0, \pi_0 \in W^{1,\infty}(\Omega), \quad \rho_0, \pi_0 \geq 0,$$

$$(8.46) \quad \sigma^*, \nu^* \in Lip^{unif}_{loc}(\Gamma_-), \quad \sigma^*, \nu^* \geq 0, \quad \sigma^*(\cdot, m, \cdot) = \nu^*(\cdot, m, \cdot) = 0 \quad \forall m \notin [m_a, M^*],$$

where $0 < m_a < M^*$. Moreover, according to the physical model introduced in [28], we also assume

$$(8.47) \quad s_j, \bar{s}_j, g_a, g_j, \pi_j, K_{ji} \in W^{1,\infty}(\mathbb{R}, \mathbb{R}_+), \quad \beta_j, Z_{ls} \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}_+),$$

$$\begin{aligned} n_j &\in L^1(\mathbb{R}, \mathbb{R}_+), \quad N^* \in W^{1,\infty}((0, t_1) \times \Omega, \mathbb{R}_+), \\ \text{supp}(g_{ol}), \text{supp}(g_j) &\subseteq [m_a, M_a], \quad \text{supp}(s_j), \text{supp}(\bar{s}_j) \subseteq [m_a, M^*], \\ \beta_j(m', m'') &= Z_{ij}(m', m'') = 0, \quad m' + m'' \geq M^*, \end{aligned}$$

with $(j, i) \in \{(l, s), (s, l)\}$ and $0 < m_a < M_a < M^*$.

We can now present a theorem of well-posedness about IBVP (8.1)-(8.8)

Theorem 8.1. *Assume that the hypotheses (8.37)-(8.47) are verified. Then there exists $0 < t^* \leq t_1$ such that (8.30)-(8.33) admits one and only one solution $(\rho, \pi, \sigma, \nu) \in W^{1,\infty}(Q_{t^*})^2 \times W^{1,\infty}(S_{t^*})^2$; furthermore, ρ, π, σ, ν take non-negative values and satisfy the relation*

$$(8.48) \quad \sigma(\cdot, m, \cdot) = \nu(\cdot, m, \cdot) = 0 \quad \forall m \notin [m_a, M^*].$$

The function (ρ, π, σ, ν) is also said to be the generalized solution for the equations (8.1)-(8.4) with the initial conditions (8.5)-(8.6) and the boundary conditions (8.7)-(8.8).

Moreover, for any sufficiently small t , the mapping $(\rho_0, \pi_0, \sigma^*, \nu^*, T, v, u, w, \nabla_x \cdot v, \nabla_{(x,m)} \cdot u, \nabla_{(x,m)} \cdot w) \in W^{1,\infty}(\Omega)^2 \times Lip_{loc}^{unif}(\Gamma_-)^2 \times [L^\infty(0, t; L^\infty(\Omega)) \cap L^1(0, t; W^{1,\infty}(\Omega))]^4 \times \tilde{L}_{x_3}^-(S_t)^2 \times [L^\infty(0, t; L^\infty(\Omega)) \cap L^1(0, t; W^{1,\infty}(\Omega))] \times [L^\infty(0, t; L^\infty(\Omega_M)) \cap L^1(0, t; W^{1,\infty}(\Omega_M))]^2 \rightarrow (\rho, \pi, \sigma, \nu) \in L^\infty(Q_t)^2 \times L^\infty(S_t)^2$ is locally Lipschitz continuous.

Proof.

To prove this theorem we observe that the IBVP (8.1)-(8.8) without integral terms is a particular case of (3.1)-(3.4); therefore we should only study the regularity of the integral terms that appear in the equations (8.2)-(8.4) and obtain some estimates about these integrals. Now, if we assume

$$(8.49) \quad (\bar{\rho}, \bar{\pi}, \bar{\sigma}, \bar{\nu}) \in L^\infty(0, t_1; W^{1,\infty}(\Omega))^2 \times L^\infty(0, t_1; W^{1,\infty}(\Omega_M))^2,$$

then it is not difficult to check that

$$(8.50) \quad [\bar{\pi} - \pi_l(T)] F_l(\bar{\sigma}), [\bar{\pi} - \pi_s(T)] F_s(\bar{\nu}) \in L^\infty(0, t_1; L^\infty(\Omega)) \cap L^1(0, t_1; W^{1,\infty}(\Omega)),$$

$$S_a(\bar{\pi}, \bar{\sigma}, \bar{\nu}), S_q(\bar{\sigma}, \bar{\nu}), N_q(\bar{\sigma}, \bar{\nu}) \in L^\infty(0, t_1; L^\infty(\Omega_M)) \cap L^1(0, t_1; W^{1,\infty}(\Omega_M)).$$

Now, we have the following useful estimates, for example, about $Q_l(\bar{\sigma}, \bar{\sigma})$

$$(8.51) \quad \|Q_l(\bar{\sigma}, \bar{\sigma})\|_{L^1(0,t;L^\infty(G))} \leq Ct \|\bar{\sigma}\|_{L^\infty(S_t)}^2,$$

$$\|\nabla_{(m,x)} Q_l(\bar{\sigma}, \bar{\sigma})\|_{L^1(0,t;L^\infty(G))} \leq Ct(\|\bar{\sigma}\|_{L^\infty(S_t)}^2 + \|\nabla_{(m,x)} \bar{\sigma}\|_{L^\infty(S_t)}^2), \quad 0 < t \leq t_1.$$

Moreover, assuming $\bar{\sigma}^{(k)} \in L^\infty(0, t_1; W^{1,\infty}(\Omega_M))$ ($k = 1, 2$), we deduce

$$(8.52) \quad \begin{aligned} & \|Q_l(\bar{\sigma}^{(2)}, \bar{\sigma}^{(2)}) - Q_l(\bar{\sigma}^{(1)}, \bar{\sigma}^{(1)})\|_{L^1(0,t;L^\infty(G))} \leq \\ & \leq Ct(\|\bar{\sigma}^{(1)}\|_{L^\infty(S_t)} \vee \|\bar{\sigma}^{(2)}\|_{L^\infty(S_t)}) \|\bar{\sigma}^{(1)} - \bar{\sigma}^{(2)}\|_{L^\infty(S_t)}. \end{aligned}$$

In a similar way, we obtain analogous estimates for the other integral terms.

After having obtained these regularity results and estimates for the integral terms, we understand that it is not so hard to extend the proof of the Theorem 3.1 to prove Theorem 8.1.

□

Références

- [1] Ambrosio L. : *Transport equations and Cauchy problem for BV vector fields*. Invent. Math., 158 n°2 (2004), pp. 227-260.
- [2] Ascoli D., Selvaduray Steave C. : *Wellposedness in the Lipschitz class for a hyperbolic system arising from a model of the atmosphere including water phase transitions*. Nonlinear Differ. Equ. Appl. 21 (2014), pp. 263-287.
- [3] Bardos C. : *Problème aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d'approximations; application à l'équation de transport*. Ann. Scient. Ec. Norm. Sup. 4ième série 3, pp. 185-233 (1970).
- [4] Belhireche, H., Aissaoui, M.Z., Fujita Yashima, H. : *Equations monodimensionnelles du mouvement de l'air avec la transition de phase de l'eau*. Sci. Techn. Univ. Constantine - A, Vol. 31 (2011), pp. 157-188.
- [5] Belhireche H., Zine M., Fujita Yashima H. : *Solution globale de l'équation de coagulation de gouttelettes en chute*. Quaderno Dip. Mat. Univ. Torino n.5/2012.
- [6] Belhireche H., Selvaduray S. C. : *Global solution for the coagulation equation of water droplets in atmosphere between two horizontal planes*. (In preparation)
- [7] Benzoni-Gavage, S., Serre, D. : *Multi-dimensional Hyperbolic Partial Differential Equations*. Oxford science publications 2007.
- [8] Besson, O., J., Pousin : *Solutions for linear conservation laws with velocity field in L^∞* . Arch. Rational Mech. Anal 186 (2007), pp.159-175.
- [9] Boyer, F. : *Trace theorems and spatial continuity properties for the solutions of the transport equation*. Differential Integral Equations 18(2005), no.8, pp.891-934.
- [10] Brandi, P., Salvadori, A., Kamont, Z. : *Existence of generalized solutions of hyperbolic functional differential equations*. Nonlinear Analysis 50 (2002) pp. 919-940.
- [11] Bressan A., Serre D., Williams M., Zumbrun K., Marcati P. : *Hyperbolic systems of balance laws*. LNM 1911, Springer, 2003.
- [12] Cesari L. : *A boundary-value problem for quasilinear hyperbolic systems*. Riv. Nat. Univ. Parma, Vol. 3 (1974), pp. 107-131, 1974.
- [13] Cesari L. : *A boundary-value problem for quasilinear hyperbolic systems*,. Ann. Scuola Norm. Sup. Pisa, Vol. 4 (1974), pp. 311-358.
- [14] Cinquini-Cibrario M. : *Sistemi di equazioni a derivate parziali in più variabili indipendenti*. Ann. di Mat (IV), XLVI (1957), pp. 357-418.
- [15] Cinquini-Cibrario M. : *Ulteriori ricerche intorno ai sistemi di equazioni a derivate parziali in più variabili indipendenti*. Ann. della Scuola Normale Sup. di Pisa, tome 13 (1959), pp. 449-488.
- [16] Cinquini-Cibrario M., Cinquini S. : *Equazioni a derivate parziali di tipo iperbolico*. Edizioni Cremonese (CNR), Roma, 1964.
- [17] Cinquini-Cibrario M. : *Ulteriori risultati per sistemi di equazioni quasi lineari a derivate parziali in più variabili indipendenti*. Ist. Lombardo Accad. Sci. Lett. Rend. A 103 (1969), pp. 373-407.
- [18] Crippa, G., Donadello, C., Spinolo, L., V. : *Initial-boundary value problems for continuity equations with BV coefficients*. <http://arxiv.org/abs/1304.0975v1>, pp. 1-25.

- [19] DiPerna R.J., S., Lions P., L. : *Ordinary differential equations, transport theory and Sobolev spaces*. Invent. Math., 98(3) (1998), pp. 511-547.
- [20] Evans, L., C. : *Partial differential equations*. Graduate studies in mathematics, Volume 19, AMS, 1997.
- [21] Filippov, A.E. : *Differential equations with discontinuous righthand sides*. Kluwer Academic Publishers, 1988.
- [22] Fujita Yashima, H., Campana, V., Aissaoui, M. Z. : *Système d'équations d'un modèle du mouvement de l'air impliquant la transition de phase de l'eau dans l'atmosphère*. to appear on Ann. Math. Afr., vol.2 (2011), pp. 66-92.
- [23] Hille, E. : *Lectures on ordinary differential equations*. Addison-Wesley publishing company, 1969.
- [24] Jeffrey A. : *Quasilinear hyperbolic systems and waves* . Pitman publishing limited , 1976.
- [25] Leoni, G. : *A first course in Sobolev spaces*. Graduate studies in mathematics, Volume 105, AMS, 2009.
- [26] Merad M., Belhireche H., Fujita Yashima H. : *Solution Stationnaire de l'équation de coagulation de gouttelettes en chute avec le vent horizontal..* To appear on Rend. Sem. Mat. Univ. Padova.
- [27] Myshkis, A.D. : *On Quasilinear Generalized Canonical Hyperbolic Systems of First-Order Partial Differential Equations*. Mathematical Notes, vol.72, no. 5 (2002), pp.672-681, Translated from Matematicheskie Zametki, vol.72, no.5 (2002), pp.729-738.
- [28] Selvaduray, S., Fujita Yashima H. : *Equazioni del moto dell'aria con la transizione di fase dell'acqua nei tre stati : gassoso, liquido e solido*. Memorie della classe di scienze Fisiche, Matematiche e Naturali, Serie V, Volume 35, pp.37-69, Accademia delle Scienze di Torino, 2011. (see also <http://www.dm.unito.it/quadernidipartimento/2010/pdf/q16-10.pdf>)
- [29] Selvaduray S. C. : *On a quasilinear hyperbolic system relative to an atmospheric model on the transition of water defined on the whole space*. Journal of Mathematics and System Science, Volume 4, Number 9, September 2014, pp. 637-655.
- [30] Selvaduray, S. : *On a model about the motion of the air and the phase transitions of water in the atmosphere*. Phd Thesis, 2014.
- [31] Volpato, M. : *Sulla derivabilità, rispetto a valori iniziali ed a parametri, delle soluzioni dei sistemi di equazioni differenziali ordinarie del primo ordine*. Rendiconti del Seminario Matematico della Università di Padova, tome 28 (1958), p.71-106.
- [32] Volpato, M. : *Sul problema di Cauchy per una equazione lineare alle derivate parziali del primo ordine*. Rendiconti del Seminario Matematico della Università di Padova, tome 28 (1958), p.153-187.
- [33] Volpato, M. : *Sul problema di Cauchy per equazioni differenziali quasi lineari alle derivate parziali del primo ordine*. Rendiconti del Seminario Matematico della Università di Padova, tome 28 (1958), p.244-262.